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Impulse control problem on finite horizon with execution delay

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Abstract

We consider impulse control problems in finite horizon for diffusions with decision lag and execution delay. The new feature is that our general framework deals with the important case when several consecutive orders may be decided before the effective execution of the first one. This is motivated by financial applications in the trading of illiquid assets such as hedge funds. We show that the value functions for such control problems satisfy a suitable version of dynamic programming principle in finite dimension, which takes into account the past dependence of state process through the pending orders. The corresponding Bellman partial differential equations (PDE) system is derived, and exhibit some peculiarities on the coupled equations, domains and boundary conditions. We prove a unique characterization of the value functions to this nonstandard PDE system by means of viscosity solutions. We then provide an algorithm to find the value functions and the optimal control. This easily implementable algorithm involves backward and forward iterations on the domains and the value functions, which appear in turn as original arguments in the proofs for the boundary conditions and uniqueness results.

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Keywords: Impulse control; Execution delay; Diffusion processes; Dynamic programming; Viscosity solutions; Comparison principle

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1. Introduction

In this paper, we consider a general impulse control problem in finite horizon of a diffusion process X, with intervention lag and execution delay. This means that we may intervene on the diffusion system at any times τ_i separated at least by some fixed positive lag h, by giving some impulse ξ_i based on the information at τ_i. However, the execution of the impulse decided at τ_i is carried out with delay mh, $m \geq 1$, i.e. it is implemented at time $\tau_i + mh$, moving the system from $X_{(\tau_i+mh)^-}$ to $\Gamma(X_{(\tau_i+mh)^-}, \xi_i)$. The objective is to maximize over impulse controls $(\tau_i, \xi_i)_i$ the expected total profit on finite horizon T, of the form

$$\mathbb{E}\left[\int_0^T f(X_t)\mathrm{d}t + g(X_T) + \sum_{\tau_i+mh\leq T} c(X_{(\tau_i+mh)^-}, \xi_i)\right].$$

Such formulations appear naturally in decision-making problems in economics and finance. In many situations, firms or investors face regulatory delays (delivery lag), which may be significant, and thus need to be taken into account when management strategies are decided in an uncertain environment. Problems where firm's investment are subject to delivery lag can be found in the real options literature, for example in [2,1]. In financial market context, execution delay is related to liquidity risk (see e.g. [15]), and occurs with transaction, which requires heavy preparatory work as for hedge funds. Indeed, hedge funds frequently hold illiquid assets, and need some time to find a counterpart to buy or sell them. Furthermore, this notice period gives the hedge fund manager a reasonable investment horizon.

From a mathematical viewpoint, it is well known that impulse control problems without delay, i.e. $m = 0$, lead to variational partial differential equations (PDE), see e.g. the books [5,11]. Impulse control problems in the presence of delay were studied in [14] for $m = 1$, that is when no more than one pending order is allowed at any time. In this case, it is shown that the delay problem may be transformed into a no-delay impulse control problem. The paper [4] also considers the case $m = 1$, but when the value of the impulse is chosen at the time of execution, and on infinite horizon, and these two conditions are crucial in the proposed probabilistic resolution. We mention also the works [3] and recently [12], which study impulse problems in infinite horizon with arbitrary number of pending orders, but under restrictive assumptions on the controlled state process, like (geometric) Lévy process for X and (multiplicative) additive intervention operator Γ. In this case, the problem is reduced to a finite-dimensional one where the value functions with pending orders are directly related to the value function without order.

The main contribution of this paper is to provide a theory of impulse control problems with delay on finite horizon in a fairly general diffusion framework that deals with the important case in applications when the number of pending orders is finite, but not restricted to one, i.e. $m \geq 1$. Our chief goal is to obtain a unique tractable PDE characterization of the value functions for such problems. As usual in stochastic control problems, the first step is the derivation of a dynamic programming principle (DPP). We show a suitable version of DPP, which takes into account the past dependence of the controlled diffusion via the finite number of pending orders. The corresponding Bellman PDE system reveals some nonstandard features both on the form of the differential operators and their domains, and on the boundary conditions. Following the standard approach to stochastic control, we prove that the value functions are viscosity solutions to this Bellman PDE system, and we also state comparison principles, which allows us to obtain a unique PDE characterization. From this PDE representation, we provide an easily implemented algorithm to compute the value functions, and the optimal impulse control as byproducts. In this algorithm, we use forward and backward iterations on the value functions and on the domains, and these iterations turn out to be original arguments in the proofs for the boundary conditions and comparison principles.

The rest of the paper is organized as follows. In Section 2, we formulate the control problem and introduce the associated value functions. Section 3 deals with the dynamic programming principle in this general framework. We then state in Section 4 the unique PDE viscosity characterization for the value functions. In Section 5, we provide an algorithm for computing the value functions and the optimal impulse control. Section 6 is devoted to the proofs for the viscosity properties, and Section 7 collects the proofs for the uniqueness and comparison results.

2. Problem formulation

2.1. The control problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and $W = (W_t)_{t\geq 0}$ a standard n-dimensional Brownian motion.

An impulse control is a double sequence $\alpha = (\tau_i, \zeta_i)_{i\geq 1}$, where (τ_i) is an increasing sequence of $\mathbb{F}$-stopping times, and ξ_i are $\mathcal{F}_{\tau_i}$-measurable random variables valued in E. We require that $\tau_{i+1} - \tau_i \geq h$ a.s., where $h > 0$ is a fixed time lag between two decision times, and we assume that E, the set of impulse values, is a compact subset of $\mathbb{R}^q$. We denote by $\mathcal{A}$ this set of impulse controls.

In the absence of impulse executions, the system valued in $\mathbb{R}^d$ evolves according to:

$$\mathrm{d}X_s = b(X_s)\mathrm{d}s + \sigma(X_s)\mathrm{d}W_s, \tag{2.1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma \to \mathbb{R}^{d\times n}$ are Borel functions on $\mathbb{R}^d$, satisfying usual Lipschitz conditions. The interventions are decided at times τ_i with impulse values ξ_i based on the information at these dates, however they are executed with delay at times $\tau_i + mh$, moving the system from $X_{(\tau_i+mh)^-}$ to $X_{(\tau_i+mh)} = \Gamma(X_{(\tau_i+mh)^-}, \xi_i)$. Here Γ is a mapping from $\mathbb{R}^d \times E$ into $\mathbb{R}^d$, and we assume that Γ is continuous, and satisfies the linear growth condition:

$$\sup_{(x,e)\in\mathbb{R}^d\times E} \frac{|\Gamma(x,e)|}{1+|x|} < \infty. \tag{2.2}$$

Given an impulse control $\alpha = (\tau_i, \xi_i)_{i\geq 1} \in \mathcal{A}$, and an initial condition $X_0 \in \mathbb{R}^d$, the controlled process X^α is then defined as the solution to the s.d.e.:

$$X_s = X_0 + \int_0^s b(X_u)\mathrm{d}u + \int_0^s \sigma(X_u)\mathrm{d}W_u + \sum_{\tau_i+mh\leq s} \left(\Gamma(X_{(\tau_i+mh)^-}, \xi_i) - X_{(\tau_i+mh)^-}\right). \tag{2.3}$$

We now fix a finite horizon $T < \infty$, and in order to avoid trivialities, we assume $T - mh \geq 0$. Using standard arguments based on Burkholder–Davis–Gundy's inequality, Gronwall's lemma and (2.2), we easily check that

$$\mathbb{E}\left[\sup_{s\leq T} |X_s^\alpha|\right] < \infty. \tag{2.4}$$

Given an impulse control $\alpha = (\tau_i, \xi_i)_{i\geq 1} \in \mathcal{A}$, we consider the total profit at horizon T, defined by:

$$\Pi(\alpha) = \int_0^T f(X_s^\alpha)\mathrm{d}s + g(X_T^\alpha) + \sum_{\tau_i + mh \leq T} c(X^\alpha_{(\tau_i+mh)^-}, \xi_i),$$

and we assume that the running profit function f, the terminal profit function g, and the executed cost function c are continuous, and satisfy the linear growth condition:

$$\sup_{(x,e)\in\mathbb{R}^d\times E} \frac{|f(x)| + |g(x)| + |c(x,e)|}{1+|x|} < \infty. \tag{2.5}$$

This ensures with (2.4) that $\Pi(\alpha)$ is integrable, and we can define the control problem:

$$V_0 = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\Pi(\alpha)\right]. \tag{2.6}$$

We also impose the following assumption:

$$g(x) \geq g\left(\Gamma(x,e)\right) + c(x,e), \quad \forall (x,e) \in \mathbb{R}^d \times E. \tag{2.7}$$

This condition economically means that a decision at time $T - mh$ induces a terminal profit, which is smaller than a no-decision at this time $T - mh$, and is thus suboptimal. Mathematically, we shall see later that the condition (2.7) is crucial for the continuity of the value function associated to our problem, see Remark 4.6. 3. Finally, notice that any intervention decided after date $T - mh$ will not influence the system and so the total profit at horizon T, and therefore, we may require w.l.o.g. that any admissible impulse control $\alpha = (\tau_i, \xi_i)_{i\geq 1} \in \mathcal{A}$ satisfies $\tau_i + mh \leq T$ for all i s.t. $\tau_i < \infty$.

Financial example

Consider a financial market consisting of a money market account yielding a constant interest rate r, and a risky asset (stock) of price process $(S_t)_t$ governed by:

$$\mathrm{d}S_t = \beta(S_t)\mathrm{d}t + \gamma(S_t)\mathrm{d}W_t.$$

We denote by Y_t the number of shares in the stock, and by Z_t the amount of money (cash holdings) held by the investor at time t. We assume that the investor can only trade discretely, and her orders are executed with delay. This is modelled through an impulse control $\alpha = (\tau_i, \xi_i)_{i\geq 1} \in \mathcal{A}$, where τ_i are the decision times, and ξ_i are the numbers of stock purchased if $\xi_i \geq 0$ or sold if $\xi_i < 0$ decided at τ_i, but executed at times $\tau_i + mh$. The dynamics of Y is then given by

$$Y_t = Y_0 + \sum_{\tau_i+mh\leq t} \xi_i,$$

which means that discrete trading $\Delta Y_t := Y_t - Y_{t^-} = \xi_i$ occur at times $s = \tau_i + mh$, $i \geq 1$. In the absence of trading, the cash holdings Z grows deterministically at rate r: $\mathrm{d}Z_t = rZ_t\mathrm{d}t$. When a discrete trading ΔY_t occurs, this results in a variation of cash holdings by $\Delta Z_t := Z_t - Z_{t^-} = -(\Delta Y_t)S_t$, from the self-financing condition. In other words, the dynamics of Z is given by

$$Z_t = Z_0 + \int_0^t rZ_u\mathrm{d}u - \sum_{\tau_i+mh\leq t} \xi_i . S_{\tau_i+mh}.$$

The wealth process is equal to $L(S_t, Y_t, Z_t) = Z_t + Y_t S_t$. This financial example corresponds to the general model (2.3) with $X = (S, Y, Z)$, $b = (\beta, 0, r)'$, $\sigma = (\gamma, 0, 0)$, and $\Gamma(s, y, z, e) = (s, e, z - es)'$. In this case, condition (2.7) is satisfied with an equality. Fix now some contingent claim characterized by its payoff at time T: $H(S_T)$ for some measurable function H. The following hedging and valuation criterion is very popular in finance, and may be embedded in our general framework:

• *Utility indifference price*. Given an utility function U for the investor, an initial capital z in cash, zero in stock, and $\kappa \geq 0$ units of contingent claims, define the expected utility under optimal trading

$$V_0(z, \kappa) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[U\left(L(S_T, Y_T, Z_T) - \kappa H(S_T)\right)\right].$$

The utility indifference ask price $\pi_a(\kappa, z)$ is the price at which the investor is indifferent (in the sense that her expected utility is unchanged under optimal trading) between paying nothing and not having the claim, and receiving $\pi_a(\kappa, z)$ now to deliver κ units of claim at time T. It is then defined as the solution to: $V_0(z + \pi_a(\kappa, z), \kappa) = V_0(z, 0)$.

2.2. *Value functions*

In order to provide an analytic characterization of the control problem (2.6), we need as usual to extend the definition of this control problem to general initial conditions. However, in contrast with classical control problems without execution delay, the diffusion process solution to (2.3) is not Markovian. Actually, given an impulse control, we see that the state of the system is not only defined by its current state value at time t but also by the pending orders, that is the orders not yet executed, i.e. decided between time $t - mh$ and t. Notice that the number of pending orders is less than or equal to m. Let us then introduce the following definitions and notations. For any $t \in [0, T]$, $k = 0, \ldots, m$, we denote by

$$\begin{aligned} P_t(k) = \{&p = (t_i, e_i)_{1 \leq i \leq k} \in ([0, T - mh] \times E)^k : t_i - t_{i-1} \geq h, i = 2, \ldots, k, \\ &t - mh < t_i \leq t, i = 1, \ldots, k\}, \end{aligned}$$

the set of k pending orders not yet executed before time t, with the convention that $P_t(0) = \emptyset$. For any $p = (t_i, e_i)_{1 \leq i \leq k} \in P_t(k)$, $t \in [0, T]$, $k = 0, \ldots, m$, we denote

$$\mathcal{A}_{t,p} = \left\{\alpha = (\tau_i, \xi_i)_{i \geq 1} \in \mathcal{A} : (\tau_i, \xi_i) = (t_i, e_i), i = 1, \ldots, k \text{ and } \tau_{k+1} \geq t\right\},$$

the set of admissible impulse controls with pending orders p before time t.

For any $(t, x) \in [0, T] \times \mathbb{R}^d$, $p \in P_t(k)$, $k = 0, \ldots, m$, and $\alpha \in \mathcal{A}_{t,p}$, we denote by $X^{t,x,p,\alpha}$ the solution to (2.3) for $t \leq s \leq T$, with initial data $X_t = x$, and pending orders p, i.e.

$$X_s = x + \int_t^s b(X_u)\mathrm{d}u + \int_t^s \sigma(X_u)\mathrm{d}W_u + \sum_{t < \tau_i + mh \leq s} \left(\Gamma(X_{(\tau_i+mh)^-}, \xi_i) - X_{(\tau_i+mh)^-}\right).$$

Using standard arguments based on Burkholder–Davis–Gundy's inequality, Gronwall's lemma and (2.2), we easily check that

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |X_s^{t,x,p,\alpha}|^2\right] \leq C(1 + |x|^2), \tag{2.8}$$

for some positive constant C depending only on b, σ, Γ and T. We then consider the following performance criterion:

$$J_k(t,x,p,\alpha) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,p,\alpha})\mathrm{d}s + g(X_T^{t,x,p,\alpha}) + \sum_{t<\tau_i+mh\le T} c(X_{(\tau_i+mh)^-}^{t,x,p,\alpha},\xi_i)\right],$$

for $(t,x)\in[0,T]\times\mathbb{R}^d$, $p\in P_t(k)$, $k=0,\ldots,m$, $\alpha=(\tau_i,\xi_i)_i\in\mathcal{A}_{t,p}$, and the corresponding value functions:

$$v_k(t,x,p) = \sup_{\alpha\in\mathcal{A}_{t,p}} J_k(t,x,p,\alpha), \quad k=0,\ldots,m,\ (t,x,p)\in\mathcal{D}_k,$$

where $\mathcal{D}_k$ is the definition domain of v_k:

$$\mathcal{D}_k = \left\{(t,x,p) : (t,x)\in[0,T)\times\mathbb{R}^d,\ p\in P_t(k)\right\}.$$

For $k=0$, $P_t(0)=\emptyset$, and we write by convention $v_0(t,x)=v_0(t,x,\emptyset)$, $\mathcal{D}_0=[0,T)\times\mathbb{R}^d$ so that the original control problem in (2.6) is given by $V_0=v_0(0,X_0)$. Note, however, that v_0 is defined on $[0,T]\times\mathbb{R}^d$. Notice from (2.5) and (2.8) that the functions v_k satisfy the linear growth condition on $\mathcal{D}_k$:

$$\sup_{(t,x,p)\in\mathcal{D}_k} \frac{|v_k(t,x,p)|}{1+|x|} < \infty, \quad k=0,\ldots,m. \tag{2.9}$$

3. Dynamic programming

In this section, we state the dynamic programming relation on the value functions of our control problem with delay execution. For any $t\in[0,T]$, $\alpha=(\tau_i,\xi_i)_{i\ge1}\in\mathcal{A}$, we denote:

$$\iota(t,\alpha) = \inf\{i\ge1 : \tau_i > t-mh\} - 1 \in \mathbb{N}\cup\{\infty\}, \tag{3.1}$$

$$k(t,\alpha) = \mathrm{card}\{i\ge1 : t-mh<\tau_i\le t\} \in \{0,\ldots,m\}, \tag{3.2}$$

$$p(t,\alpha) = (\tau_{i+\iota(t,\alpha)},\xi_{i+\iota(t,\alpha)})_{1\le i\le k(t,\alpha)} \in P_t(k(t,\alpha)). \tag{3.3}$$

Theorem 3.1. *The value functions satisfy the dynamic programming principle: for all $k=0,\ldots,m$, $(t,x,p)\in\mathcal{D}_k$,*

$$v_k(t,x,p) = \sup_{\alpha\in\mathcal{A}_{t,p}} \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{\tau_i+mh\le\theta} c(X_{(\tau_i+mh)^-}^{t,x,p,\alpha},\xi_i) + v_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha))\right], \tag{3.4}$$

where θ is any stopping time valued in $[t,T]$, possibly depending on α in (3.4). This means

(DP1) *for all $\alpha\in\mathcal{A}_{t,p}$, for all θ stopping time valued in $[t,T]$,*

$$v_k(t,x,p) \ge \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X_{(\tau_i+mh)^-}^{t,x,p,\alpha},\xi_i) + v_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha))\right]. \tag{3.5}$$

(DP2) *for all* $\varepsilon > 0$, *there exists* $\alpha \in \mathcal{A}_{t,p}$ *such that for all* θ *stopping time valued in* $[t, T]$,

$$v_k(t,x,p) - \varepsilon \le \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X_{(\tau_i+mh)^-}^{t,x,p,\alpha}, \xi_i) \right.$$
$$\left. +\ v_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha))\right]. \tag{3.6}$$

We now give an explicit consequence of the above dynamic programming that will be useful in the derivation of the corresponding analytic characterization. We introduce some additional notations. For all $t \in [0,T]$, we denote by $\mathcal{I}_t$ the set of pairs (τ, ξ) where τ is a stopping time, $t \le \tau \le T - mh$ or $\tau = \infty$ a.s., and ξ is a $\mathcal{F}_\tau$-measurable random variable valued in E. For any $p = (t_i, e_i)_{1\le i\le k} \in P_t(k)$, we denote $p_- = (t_i, e_i)_{2\le i\le k}$ with the convention that $p_- = \emptyset$ when $k = 1$.

When no impulse control is applied to the system, we denote by $X_s^{t,x,0}$ the solution to (2.1) with initial data $X_t = x$, and by $\mathcal{L}$ the associated infinitesimal generator:

$$\mathcal{L}\varphi = b(x).D_x\varphi + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x)D_x^2\varphi).$$

If $t \le T - mh$, we partition, for $k \in \{1,\ldots,m\}$, the set $P_t(k)$ into $P_t(k) = P_t^1(k) \cup P_t^2(k)$ where

$$P_t^1(k)(\text{resp. } P_t^2(k)) = \left\{p = (t_i,e_i)_{1\le i\le k} \in P_t(k) : t_k > (\text{resp. } \le) t - h\right\}.$$

Else if $t \ge T - mh$, we denote $P_t^1(k) = P_t(k)$ and $P_t^2(k) = \emptyset$. We easily see from the lag constraint on the pending orders that $P_t^2(k) = \emptyset$ if $k = m$, and so $P_t(m) = P_t^1(m)$.

Corollary 3.2. *Let* $(t,x) \in [0,T) \times \mathbb{R}^d$.
(1) *For* $k \in \{1,\ldots,m\}$, *and* $p = (t_i,e_i)_{1\le i\le k} \in P_t^1(k)$, *we have for any stopping time* θ *valued in* $[t, (t_k+h) \wedge (t_1+mh))$:

$$v_k(t,x,p) = \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,0})\mathrm{d}s + v_k(\theta, X_\theta^{t,x,0}, p)\right]. \tag{3.7}$$

(2) *For* $k \in \{0,\ldots,m-1\}$, *and* $p = (t_i,e_i)_{1\le i\le k} \in P_t^2(k)$, *with the convention that* $P_t^2(k) = \emptyset$ *and* $t_1 + mh = T$ *when* $k = 0$, *we have for any stopping time* θ *valued in* $[t, (t_1+mh) \wedge (t+h))$:

$$v_k(t,x,p) = \sup_{(\tau,\xi)\in\mathcal{I}_t} \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,0})\mathrm{d}s + v_k(\theta, X_\theta^{t,x,0}, p)1_{\theta<\tau}\right.$$
$$\left. +\ v_{k+1}(\theta, X_\theta^{t,x,0}, p \cup (\tau,\xi))1_{\tau\le\theta}\right]. \tag{3.8}$$

Interpretation and remarks

(1) $P_t^1(k)$ represents the set of k pending orders where the last order is within the period $(t-h, t]$ of nonintervention before t. Hence, from time t and until time $(t_k+h) \wedge (t_1+mh)$, we cannot intervene on the diffusion system and no pending order will be executed during this time period. This is mathematically formalized by relation (3.7).

(2) $P_t^2(k)$ represents the set of k pending orders where the last order is out of the period of nonintervention before t. Hence, at time t, one has two possible decisions: either one lets continue the system or one immediately intervene. In this latter case, this order adds to the previous ones. The mathematical formalization of these two choices is translated into relation (3.8).

We now turn to the proof of the dynamic programming principle in Theorem 3.1.

3.1. Proof of dynamic programming principle

From the dynamics (2.3) of the controlled process, we derive easily the following properties (recall the notations (3.1)–(3.3)):

• Markov property of the pair $(X^\alpha, p(.,\alpha))$ for any $\alpha \in \mathcal{A}$, in the sense that

$$\mathbb{E}\left[\varphi(X^\alpha_{\theta_2})\middle|\mathcal{F}_{\theta_1}\right] = \mathbb{E}\left[\varphi(X^\alpha_{\theta_2})\middle|(X^\alpha_{\theta_1}, p(\theta_1,\alpha))\right],$$

for any bounded measurable function φ, and stopping times $\theta_1 \le \theta_2$ a.s.

• Causality of the control, in the sense that for any $\alpha = (\tau_i, \xi_i)_{i\ge1} \in \mathcal{A}$, and θ stopping time,

$$\alpha^\theta \in \mathcal{A}_{\theta, p(\theta,\alpha)}, \quad \text{and} \quad p(\theta,\alpha) \in k(\theta,\alpha) \quad a.s.$$

where we set $\alpha^\theta = (\tau_{i+\iota(\theta,\alpha)}, \xi_{i+\iota(\theta,\alpha)})_{i\ge1}$.

• Pathwise uniqueness of the state process,

$$X^{t,x,p,\alpha} = X^{\theta, X^{t,x,p,\alpha}_\theta, p(\theta,\alpha), \alpha^\theta} \quad \text{on } [\theta, T],$$

for any $(t,x,p) \in \mathcal{D}_k$, $k = 0,\ldots,m$, $\alpha \in \mathcal{A}_{t,p}$, and $\theta \in \mathcal{T}_{t,T}$ the set of stopping times valued in $[t,T]$.

From the above properties, we deduce by usual arguments the assertion (DP2) of the dynamic programming principle in Theorem 3.1, which can be formulated equivalently in

Proposition 3.3. (DP2) *For all $k = 0,\ldots,m$, $(t,x,p) \in \mathcal{D}_k$, we have*

$$v_k(t,x,p) \le \sup_{\alpha\in\mathcal{A}_{t,p}} \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(X^{t,x,p,\alpha}_s)\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-}, \xi_i) + v_{k(\theta,\alpha)}(\theta, X^{t,x,p,\alpha}_\theta, p(\theta,\alpha))\right].$$

Proof. Fix $(t,x,p) \in \mathcal{D}_k$, $k = 0,\ldots,m$, and take arbitrary $\alpha \in \mathcal{A}_{t,p}$, $\theta \in \mathcal{T}_{t,T}$. From the definitions of the performance criterion and the value functions, the law of iterated conditional expectations, Markov property, pathwise uniqueness, and causality features of our model, we get the successive relations

$$\begin{aligned} J_k(t,x,p,\alpha) = \mathbb{E}\Bigg[&\int_t^\theta f(X^{t,x,p,\alpha}_s)\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-}, \xi_i) \\ &+\mathbb{E}\Bigg[\int_\theta^T f(X^{t,x,p,\alpha}_s)\mathrm{d}s + g(X^{t,x,p,\alpha}_T) \\ &+ \sum_{\theta<\tau_i+mh\le T} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-}, \xi_i)\Bigg|\mathcal{F}_\theta\Bigg]\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg[\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-},\xi_i) \\
&\quad + J_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha), \alpha^\theta)\Bigg] \\
&\le \mathbb{E}\Bigg[\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-},\xi_i) \\
&\quad + v_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha))\Bigg].
\end{aligned}
$$

Since θ and α are arbitrary, we obtain the required inequality. □

As usual, the assertion (DP1) of the dynamic programming principle in Theorem 3.1 requires in addition to the Markov, causality and pathwise uniqueness properties, a measurable selection theorem. This can be formulated equivalently in

Proposition 3.4. (DP1) *For all $k = 0, \ldots, m$, $(t, x, p) \in \mathcal{D}_k$, we have*

$$
\begin{aligned}
v_k(t,x,p) \ge \sup_{\alpha\in\mathcal{A}_{t,p}} \sup_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\Bigg[&\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-},\xi_i) \\
&+ v_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha))\Bigg].
\end{aligned}
$$

Proof. Fix $(t, x, p) \in \mathcal{D}_k$, $k = 0, \ldots, m$, and arbitrary $\alpha \in \mathcal{A}_{t,p}$, $\theta \in \mathcal{T}_{t,T}$. By definition of the value functions, for any $\varepsilon > 0$ and $\omega \in \Omega$, there exists $\alpha_{\varepsilon,\omega} \in \mathcal{A}_{\theta(\omega),p(\theta(\omega),\alpha(\omega))}$, which is an ε-optimal control for $v_{k(\theta(\omega),\alpha(\omega))}$ at $(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha))(\omega)$. By a measurable selection theorem (see e.g. Chapter 7 in [6]), there exists $\bar\alpha_\varepsilon \in \mathcal{A}_{\theta,p(\theta,\alpha)}$ s.t. $\bar\alpha_\varepsilon(\omega) = \alpha_{\varepsilon,\omega}(\omega)$ a.s., and so

$$
v_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha)) - \varepsilon \le J_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha), \bar\alpha_\varepsilon) \quad a.s. \tag{3.9}
$$

Now, we define by concatenation the impulse control $\bar\alpha$ consisting of the impulse control components of α until (including eventually) time τ, and the impulse control components of $\bar\alpha_\varepsilon$ strictly after time τ. By construction, $\bar\alpha \in \mathcal{A}_{t,p}$, $X^{t,x,p,\bar\alpha} = X^{t,x,p,\alpha}$ on $[t,\theta]$, $k(\theta,\bar\alpha) = k(\theta,\alpha)$, $p(\theta,\bar\alpha) = p(\theta,\alpha)$, and $\bar\alpha^\theta = \bar\alpha_\varepsilon$. Hence, similarly as in Proposition 3.3, by using law of iterated conditional expectations, Markov property, pathwise uniqueness, and causality features of our model, we get

$$
\begin{aligned}
J_k(t,x,p,\bar\alpha) = \mathbb{E}\Bigg[&\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{t<\tau_i+mh\le\theta} c(X^{t,x,p,\alpha}_{(\tau_i+mh)^-},\xi_i) \\
&+ J_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha), \bar\alpha_\varepsilon)\Bigg].
\end{aligned}
$$

Together with (3.9), this implies

$$v_k(t,x,p) \geq J_k(t,x,p,\bar{\alpha}) \geq \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,p,\alpha})\mathrm{d}s + \sum_{t<\tau_i+mh\leq\theta} c(X_{(\tau_i+mh)^-}^{t,x,p,\alpha}, \xi_i) \right.$$
$$\left. + \ v_{k(\theta,\alpha)}(\theta, X_\theta^{t,x,p,\alpha}, p(\theta,\alpha))\right] - \varepsilon.$$

From the arbitrariness of ε, α, and θ, this proves the required result. □

We end this paragraph by proving Corollary 3.2.

Proof of Corollary 3.2. (1) Fix $k \in \{1,\ldots,m\}$, $(t,x) \in [0,T]\times\mathbb{R}^d$, $p = (t_i,e_i)_{1\leq i\leq k} \in P_t^1(k)$ such that $t_1 + mh \leq T$, and θ stopping time valued in $[t, (t_k+h)\wedge(t_1+mh))$. Then, we observe that for all $\alpha = (\tau_i,\xi_i)_{i\geq1} \in \mathcal{A}_{t,p}$, $X^{t,x,p,\alpha} = X^{t,x,0}$ on $[t,\theta]$, $\tau_i + mh > \theta$, $k(\theta,\alpha) = k$, and $p(\theta,\alpha) = p$ a.s. Hence, relation (3.7) follows immediately from (3.4).
(2) For $k \in \{0,\ldots,m-1\}$, $p = (t_i,e_i)_{1\leq i\leq k} \in P_t^2(k)$ such that $t_1 + mh \leq T$, and θ stopping time valued in $[t, (t_1+mh)\wedge(t+h))$. Let $\alpha = (\tau_i,\xi_i)_{i\geq1}$ be some arbitrary element in $\mathcal{A}_{t,p}$, and set $\tau = \tau_{k+1}$, $\xi = \xi_{k+1}$. Notice that $(\tau,\xi) \in \mathcal{I}_t$. Then, we see that $X^{t,x,p,\alpha} = X^{t,x,0}$ on $[t,\theta]$, $\tau_i + mh > \theta$, $k(\theta,\alpha) = k$, $p(\theta,\alpha) = p$ if $\theta < \tau$, and $k(\theta,\alpha) = k+1$, $p(\theta,\alpha) = p \cup (\tau,\xi)$ if $\theta \geq \tau$. We deduce from (3.5) that

$$v_k(t,x,p) \geq \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,0})\mathrm{d}s + v_k(\theta, X_\theta^{t,x,0}, p)1_{\theta<\tau}\right.$$
$$\left. + \ v_{k+1}(\theta, X_\theta^{t,x,0}, p\cup(\tau,\xi))1_{\tau\leq\theta}\right],$$

and this inequality holds for any $(\tau,\xi) \in \mathcal{I}_t$ by arbitrariness of α. Furthermore, from (3.6), for all $\varepsilon > 0$, there exists $(\tau,\xi) \in \mathcal{I}_t$ s.t.

$$v_k(t,x,p) - \varepsilon \leq \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,0})\mathrm{d}s + v_k(\theta, X_\theta^{t,x,0}, p)1_{\theta<\tau}\right.$$
$$\left. + \ v_{k+1}(\theta, X_\theta^{t,x,0}, p\cup(\tau,\xi))1_{\tau\leq\theta}\right].$$

The two previous inequalities give the required relation

$$v_k(t,x,p) = \sup_{(\tau,\xi)\in\mathcal{I}_t} \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,0})\mathrm{d}s + v_k(\theta, X_\theta^{t,x,0}, p)1_{\theta<\tau}\right.$$
$$\left. + \ v_{k+1}(\theta, X_\theta^{t,x,0}, p\cup(\tau,\xi))1_{\tau\leq\theta}\right].$$

In the next sections, we show how one can exploit these dynamic programming relations in order to characterize analytically the value functions by means of partial differential equations, and then to provide an algorithm for computing the value functions. □

4. PDE system viscosity characterization

For $k = 1,\ldots,m$, let us introduce the subspace Θ_k of $[0, T-mh]^k$:

$$\Theta_k = \left\{t^{(k)} = (t_i)_{1\leq i\leq k} \in [0,T-mh]^k : t_k - t_1 < mh, t_i - t_{i-1} \geq h, i = 2,\ldots,k\right\}.$$

We shall write, by misuse of notation, $p = (t_i,e_i)_{1\leq i\leq k} = (t^{(k)}, e^{(k)})$, for any $t^{(k)} = (t_i)_{1\leq i\leq k} \in \Theta_k$, $e^{(k)} = (e_i)_{1\leq i\leq k} \in E^k$. By convention, we set $\Theta_k = E^k = \emptyset$ for $k = 0$. Notice that for all

$t \in [0, T]$, and $p = (t^{(k)}, e^{(k)}) \in \Theta_k \times E^k$, $k = 0, \ldots, m$, we have

$$p \in P_t(k) \Longleftrightarrow t \in \mathbb{T}_p(k),$$

where $\mathbb{T}_p(k)$ is the time domain in $[0, T]$ defined by:

$$\mathbb{T}_p(k) = [t_k, t_1 + mh).$$

By convention, we set $\mathbb{T}_p(k) = [0, T)$ for $k = 0$. We can then rewrite the domain $\mathcal{D}_k$ of the value function v_k in terms of union of time-space domains:

$$\mathcal{D}_k = \left\{(t, x, p) : (t, x) \in \mathbb{T}_p(k) \times \mathbb{R}^d,\ p \in \Theta_k \times E^k\right\}.$$

Therefore, the determination of the value function v_k, $k = 0, \ldots, m$, is equivalent to the determination of the function $v_k(.,., p)$ on $\mathbb{T}_p(k) \times \mathbb{R}^d$ for all $p \in \Theta_k \times E_k$. The main goal of this paper is to provide an analytic characterization of these functions by means of the dynamic programming principle stated in the previous section.

For $k = 0$, we set $\mathcal{D}_0 = [0, T) \times \mathbb{R}^d$. For $p = (t_i, e_i)_{1 \le i \le k} \in \Theta_k \times E^k$, we partition the time domain $\mathbb{T}_p(k)$ into $\mathbb{T}_p(k) = \mathbb{T}^1_p(k) \cup \mathbb{T}^2_p(k)$ where

$$\mathbb{T}^2_p(k) = \left\{t \in \mathbb{T}_p(k) \cap [0, T - mh] : t \ge t_k + h\right\} = [t_k + h, t_1 + mh) \cap [0, T - mh],$$

with the convention that $[s, t) = \emptyset$ if $s \ge t$. We then partition $\mathcal{D}_k$ into $\mathcal{D}_k = \mathcal{D}^1_k \cup \mathcal{D}^2_k$ where

$$\mathcal{D}^1_k = \left\{(t, x, p) \in \mathcal{D}_k : t \in \mathbb{T}^1_p(k)\right\}, \qquad \mathcal{D}^2_k = \left\{(t, x, p) \in \mathcal{D}_k : t \in \mathbb{T}^2_p(k)\right\}.$$

Notice that for $k = 1, \ldots, m$, and any $p \in \Theta_k \times E^k$, $\mathbb{T}^1_p(k)$ is never empty. In particular, $\mathcal{D}^1_k \neq \emptyset$. For $k = m$, and any $p = (t_i, e_i)_{1 \le i \le m} \in \Theta_m \times E^m$, we have $t_m + h \ge t_1 + mh$, and so $\mathbb{T}^2_p(m) = \emptyset$. Hence, $\mathcal{D}^2_m = \emptyset$ and $\mathcal{D}_m = \mathcal{D}^1_m$.

The PDE system to our control problem is formally derived by sending θ to $t < t_1 + mh$ into dynamic programming relations (3.7) and (3.8). This provides equations for the value functions v_k on $\mathcal{D}_k$, which take the following nonstandard form, and are divided into:

$$-\frac{\partial v_k}{\partial t}(t, x, p) - \mathcal{L}v_k(t, x, p) - f(x) = 0 \quad \text{on } \mathcal{D}^1_k,\ k = 0, \ldots, m, \tag{4.1}$$

$$\min\left\{-\frac{\partial v_k}{\partial t}(t, x, p) - \mathcal{L}v_k(t, x, p) - f(x),\right.$$
$$\left. v_k(t, x, p) - \sup_{e \in E} v_{k+1}(t, x, p \cup (t, e))\right\} = 0 \quad \text{on } \mathcal{D}^2_k,\ k = 0, \ldots, m-1, \tag{4.2}$$

with the convention that $\mathcal{D}^1_0 = (T - mh, T) \times \mathbb{R}^d$ and $\mathcal{D}^2_0 = [0, T - mh] \times \mathbb{R}^d$.

As usual, the value functions need not be smooth, and even not known to be continuous a priori, and we shall work with the notion of (discontinuous) viscosity solutions (see [7] or [9] for classical references on the subject), which we adapt in our context as follows. For a locally bounded function w_k on $\mathcal{D}_k$, we denote $\underline{w_k}$ (resp. $\overline{w}_k$) its lower-semicontinuous (resp. upper-semicontinuous) envelope, i.e.

$$\underline{w_k}(t, x, p) = \liminf_{(t', x', p') \to (t, x, p)} w_k(t', x', p'),$$
$$\overline{w_k}(t, x, p) = \limsup_{(t', x', p') \to (t, x, p)} w_k(t', x', p'), \quad (t, x, p) \in \mathcal{D}_k,\ k = 0, \ldots, m.$$

Definition 4.1. We say that a family of locally bounded functions w_k on $\mathcal{D}_k$, $k = 0, \ldots, m$, is a viscosity supersolution (resp. subsolution) of (4.1) and (4.2) on $\mathcal{D}_k$, $k = 0, \ldots, m$, if:
(i) for all $k = 1, \ldots, m$, $(t_0, x_0, p_0) \in \mathcal{D}_k^1$, and $\varphi \in C^2(\mathcal{D}_k^1)$, which realizes a local minimum of $\underline{w_k} - \varphi$ (resp. maximum of $\overline{w_k} - \varphi$), we have

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0, p_0) - \mathcal{L}\varphi(t_0, x_0) - f(x_0) \geq 0 \quad (\text{resp.} \leq 0).$$

(ii) for all $k = 0, \ldots, m-1$, $(t_0, x_0, p_0) \in \mathcal{D}_k^2$, and $\varphi \in C^2(\mathcal{D}_k^2)$, which realizes a local minimum of $\underline{w_k} - \varphi$ (resp. maximum of $\overline{w_k} - \varphi$), we have

$$\min\left\{-\frac{\partial \varphi}{\partial t}(t_0, x_0, p_0) - \mathcal{L}\varphi(t_0, x_0, p_0) - f(x_0), \right.$$
$$\left. \underline{w_k}(t_0, x_0, p_0) - \sup_{e \in E} \underline{w_{k+1}}(t_0, x_0, p_0 \cup (t_0, e))\right\} \geq 0$$

(resp.

$$\min\left\{-\frac{\partial \varphi}{\partial t}(t_0, x_0, p_0) - \mathcal{L}\varphi(t_0, x_0, p_0) - f(x_0), \right.$$
$$\left. \overline{w_k}(t_0, x_0, p_0) - \sup_{e \in E} \overline{w_{k+1}}(t_0, x_0, p_0 \cup (t_0, e))\right\} \leq 0).$$

We say that a family of locally bounded functions w_k on $\mathcal{D}_k$, $k = 0, \ldots, m$, is a viscosity solution of (4.1) and (4.2) if it is a viscosity supersolution and subsolution of (4.1) and (4.2).

We then state the viscosity property of the value functions to our control problem.

Proposition 4.2 (*Viscosity Property*). *The family of value functions v_k, $k = 0, \ldots, m$, is a viscosity solution to (4.1) and (4.2). Moreover, for all $k = 0, \ldots, m-1$, $(t, x, p) \in \mathcal{D}_k^2$, $p = (t_i, e_i)_{1 \leq i \leq k}$ with $t = t_k + h$ or $t = T - mh$, we have:*

$$\underline{v_k}(t, x, p) \geq \sup_{e \in E} \underline{v_{k+1}}(t, x, p \cup (t, e)). \tag{4.3}$$

In order to have a complete characterization of the value functions, and so of our control problem, we need to determine the suitable boundary conditions. These concern for $k = 1, \ldots, m$ the time-boundary of $\mathcal{D}_k$, i.e. the points $(t_1 + mh, x, p)$ for $x \in \mathbb{R}^d$, $p = (t_i, e_i)_{1 \leq i \leq k} \in \Theta_k \times E^k$, and also the value function v_0 on (T, x), $x \in \mathbb{R}^d$. For a locally bounded function w_k on $\mathcal{D}_k$, $k = 1, \ldots, m$, we denote

$$\overline{w_k}(t_1 + mh, x, p) = \limsup_{\substack{(t, x', p') \to (t_1 + mh, x, p) \\ (t, x', p') \in \mathcal{D}_k}} w_k(t, x', p'),$$

$$\underline{w_k}(t_1 + mh, x, p) = \liminf_{\substack{(t, x', p') \to (t_1 + mh, x, p) \\ (t, x', p') \in \mathcal{D}_k}} v_k(t, x', p'), \quad x \in \mathbb{R}^d,\ p = (t_i, e_i)_{1 \leq i \leq k} \in \Theta_k,$$

and if these two limits are equal, we set

$$w_k((t_1 + mh)^-, x, p) = \overline{w_k}(t_1 + mh, x, p) = \underline{w_k}(t_1 + mh, x, p).$$

Proposition 4.3 (*Boundary Data*). (i) *For $k = 1, \ldots, m$, $p = (t_i, e_i)_{1 \leq i \leq k} \in \Theta_k \times E^k$, $x \in \mathbb{R}^d$, $v_k((t_1 + mh)^-, x, p)$ exists and:*

$$v_k((t_1 + mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-). \tag{4.4}$$

(ii) *At time T, for all $x \in \mathbb{R}^d$, $v_0(T^-, x)$ exists and:*

$$v_0(T^-, x) = g(x). \tag{4.5}$$

We can now state the unique PDE characterization result for our control delay problem.

Theorem 4.4. *The family of value functions v_k, $k = 0, \ldots, m$, is the unique viscosity solution to (4.1) and (4.2), which satisfy (4.3), the boundary data (4.4) and (4.5), and the linear growth condition (2.9). Moreover, v_k is continuous on $\mathcal{D}_k$, $k = 0, \ldots, m$.*

Remark 4.5 (*Case $m = 1$*). In the particular case where the execution delay is equal to the intervention lag, i.e. $m = 1$, we have two value functions v_0 and v_1, and the system (4.1) and (4.2) may be significantly simplified. Actually, from the linear PDE (4.1) and the boundary data (4.4) for $k = m = 1$, we have the Feynman–Kac representation:

$$v_1(t, x, (t_1, e_1)) = \mathbb{E}\left[\int_t^{t_1+h} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t_1+h}^{t,x,0}, e) + v_0(t_1 + h, \Gamma(X_{t_1+h}^{t,x,0}, e))\right], \tag{4.6}$$

for all $(t_1, e_1) \in [0, T - h] \times E$, $(t, x) \in [t_1, t_1 + h) \times \mathbb{R}^d$. By plugging (4.6) for $t = t_1$ into (4.2) for $k = 0$, we obtain the variational inequality satisfied by v_0:

$$\begin{aligned} 0 = \min \Bigg\{ &-\frac{\partial v_0}{\partial t} - \mathcal{L}v_0 - f, \\ &v_0 - \sup_{e \in E} \mathbb{E}\left[\int_t^{t+h} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t+h}^{t,x,0}, e) + v_0(t + h, \Gamma(X_{t+h}^{t,x,0}, e))\right]\Bigg\} \\ &\text{on } [0, T - h] \times \mathbb{R}^d, \end{aligned} \tag{4.7}$$

together with the terminal condition for $k = 0$ (see (5.1)):

$$v_0(t, x) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,0})\mathrm{d}s + g(X_T^{t,x,0})\right], \quad (t, x) \in (T - h, T] \times \mathbb{R}^d. \tag{4.8}$$

Therefore, in the case $m = 1$, and as observed in [14], the original problem is reduced to a no-delay impulse control problem (4.7) for v_0, and v_1 is explicitly related to v_0 by (4.6). Eqs. (4.7) and (4.8) can be solved by iterated optimal stopping problems, see the details in the next section in the more general case $m \geq 1$.

Remark 4.6. In the general case $m \geq 1$, we point out the peculiarities of the PDE characterization for our control delay problem.

1. The dynamic programming coupled system (4.1) and (4.2) has a nonstandard form. For fixed k, there is a discontinuity on the differential operator of the equation satisfied by v_k on $\mathcal{D}_k$. Indeed, the PDE is divided into a linear equation on the subdomain $\mathcal{D}_k^1$, and a variational inequality with obstacle involving the value function v_{k+1} on the subdomain $\mathcal{D}_k^2$. Moreover, the time domain $\mathbb{T}_p(k)$ of $\mathcal{D}_k$ for $v_k(., x, p)$ depends on the argument $p \in \Theta_k$. With respect to usual comparison principle of nonlinear PDE, we state a uniqueness result for viscosity solutions satisfying in addition the inequality (4.3) at the discontinuity of the differential operator.

2. The boundary data also present some specificities. For fixed k, the condition in (4.4) concerns as usual data on the time-boundary of the domain $\mathcal{D}_k$ on which the value function v_k satisfies a PDE. However, it involves data on the value function v_{k-1}, which is a priori not known.

3. The continuity property of the value functions v_k on $\mathcal{D}_k$ is not at all obvious a priori from the very definitions of v_k, and is proved actually as consequences of comparison principles and boundary data for the system (4.1) and (4.2), see Proposition 7.8. In particular, if assumption (2.7) is relaxed, then continuity does not hold necessarily for the value function. For example, by taking $b = \sigma = f = g = 0$ and $c(x, e) = 1$ for all $(x, e) \in \mathbb{R}^d \times E$, we easily see that $v_0(t, x) = \max(0, \lceil \frac{T-mh-t}{h} \rceil)$, which is obviously not continuous.

The PDE characterization in Theorem 4.4 means that the value functions are in theory completely determined by the resolution of the PDE system (4.1) and (4.2) together with the boundary data (4.4) and (4.5). We show in the next section how to solve this system and compute in practice these value functions and the associated optimal impulse controls.

5. An algorithm to compute the value functions and the optimal control

5.1. Computation of the value functions

We first make the following observation. Let us denote by F_0 the function defined on $[0, T - mh] \times \mathbb{R}^d$ by

$$F_0(t, x) = \sup_{e \in E} v_1(t, x, (t, e)), \quad (t, x) \in \mathcal{D}_0^2$$

and on $(T - mh, T] \times \mathbb{R}^d$ by:

$$F_0(t, x) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,0})\mathrm{d}s + g(X_T^{t,x,0})\right], \quad (t, x) \in \mathcal{D}_0^1.$$

This function F_0 clearly satisfies on $\mathcal{D}_0^1$ the linear PDE: $-\frac{\partial F_0}{\partial t} - \mathcal{L}F_0 - f = 0$, together with the terminal condition $F_0(T^-, x) = g(x)$. Hence, with (4.1) for $k = 0$ and a standard uniqueness result, this shows that

$$v_0(t, x) = F_0(t, x), \quad (t, x) \in (T - mh, T] \times \mathbb{R}^d. \tag{5.1}$$

Moreover, from the PDE (4.2) for $k = 0$, and a standard uniqueness result for the corresponding free-boundary problem, we may also represent v_0 as the solution to the optimal stopping problem:

$$v_0(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}[F_0(\tau, X_\tau^{t,x,0})], \quad (t, x) \in [0, T] \times \mathbb{R}^d. \tag{5.2}$$

Hence, the value function v_0 is completely determined once we can compute v_1.

We show how one can compute $v_k(., ., p)$ on $\mathbb{T}_p(k) \times \mathbb{R}^d$ for all $p \in \Theta_k \times E^k$, $k = 1, \ldots, m$ and v_0 on $[0, T] \times \mathbb{R}^d$.

For $k = 1, \ldots, m$, and any $n \geq 1$, we denote:

$$\Theta_k(n) = \left\{t^{(k)} = (t_i)_{1 \leq i \leq k} \in \Theta_k : t_1 > T - nh\right\}, \quad N = \inf\{n \geq 1 : T - nh < 0\},$$

so that $\Theta_k(n)$ is strictly included in $\Theta_k(n + 1)$ for $n = 1, \ldots, N - 1$, and $\Theta_k(N) = \Theta_k$. We also denote for $k = 0$, and $n \geq 1$, $\mathbb{T}^n(0) = (T - nh, T] \cap [0, T]$ so that $\mathbb{T}^n(0) = (T - nh, T]$

is increasing with $n = 1, \ldots, N-1$, and $\mathbb{T}^N(0) = [0, T]$. We assumed $T - mh \geq 0$ to avoid trivialities so that $N > m$. For $n = m, \ldots, N$, we set $m(n) = (n-m) \wedge m$ the maximum number of pending orders at step n, and we denote for $k = 0, \ldots, m(n)$:

$$\mathcal{D}_k(n) = \left\{(t, x, p) \in \mathcal{D}_k : p \in \Theta_k(n) \times E^k\right\}, \quad \mathcal{D}_k^i(n) = \mathcal{D}_k(n) \cap \mathcal{D}_k^i, \quad i = 1, 2,$$

with the convention that $\mathcal{D}_0(n) = \mathbb{T}^n(0) \times \mathbb{R}^d$, so that $\mathcal{D}_k(n)$ is strictly included in $\mathcal{D}_k(n+1)$ for $n = 1, \ldots, N-1$, $\mathcal{D}_k(N) = \mathcal{D}_k$ and $m(N) = m$. We shall compute v_k on $\mathcal{D}_k(n)$, $k = 0, \ldots, m(n)$, by forward induction on $n = m, \ldots, N$ and backward induction on k.

► **Initialization phase**: $n = m$. From (4.5) and (5.1), we know the values of v_0 on $\mathcal{D}_0(m)$:

$$v_0(t, x) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,0})\mathrm{d}s + g(X_T^{t,x,0})\right].$$

► **Step** $n \to n+1$ **for** $n \in \{m, \ldots, N-1\}$. Suppose that we know the values of v_k on $\mathcal{D}_k(n)$, $k = 0, \ldots, m(n)$. In order to determine v_k on $\mathcal{D}_k(n+1)$, $k = 0, \ldots, m(n+1)$, it suffices to compute $v_k(.,.,p)$ on $\mathbb{T}_p(k) \times \mathbb{R}^d$ for all $p \in \Theta_k(n+1) \times E^k$, $k = 1, \ldots, m(n+1)$, and v_0 on $\mathbb{T}^{n+1}(0) \times \mathbb{R}^d$. We shall argue by backward induction on $k = m(n+1), \ldots, 0$.

• Let $k = m(n+1)$, and take some arbitrary $p = (t_i, e_i)_{1 \leq i \leq m(n+1)} \in \Theta_{m(n+1)}(n+1) \times E^{m(n+1)}$. Recall that $\mathbb{T}_p^2(m(n+1))$ is empty so that $\mathbb{T}_p(m(n+1)) = \mathbb{T}_p^1(m(n+1)) = [t_{m(n+1)}, t_1 + mh)$. From (4.4) for $k = m$, we have $v_{m(n+1)}((t_1 + mh)^-, x, p) = c(x, e_1) + v_{m(n+1)-1}(t_1 + mh, \Gamma(x, e_1), p_-)$ for all $x \in \mathbb{R}^d$, which is known from step n since either $p_- \in \Theta_{m(n+1)-1}(n) \times E^{m(n+1)-1}$ when $m(n+1) > 1$, or $t_1 + mh \in \mathbb{T}^n(0)$ when $m(n+1) - 1 = 0$. We then solve $v_{m(n+1)}(.,.,p)$ on $\mathbb{T}_p^1(m(n+1)) \times \mathbb{R}^d$ from (4.1) for $k = m(n+1)$, which gives:

$$\begin{aligned} v_{m(n+1)}(t, x, p) = \mathbb{E}\Bigg[&\int_t^{t_1+mh} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t_1+mh}^{t,x,0}, e_1) \\ &+ v_{m(n+1)-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\Bigg]. \end{aligned}$$

We have then computed the value of $v_{m(n+1)}(.,.,p)$ on $\mathbb{T}_p(m(n+1)) \times \mathbb{R}^d$.

• From $k+1 \to k$ for $k = m(n+1)-1, \ldots, 1$. (This step is empty when $m(n+1) = 1$). Suppose that we know the values of $v_{k+1}(.,.,p)$ on $\mathbb{T}_p(k+1) \times \mathbb{R}^d$ for all $p \in \Theta_{k+1}(n+1) \times E^{k+1}$. Take now some arbitrary $p = (t_i, e_i)_{1 \leq i \leq k} \in \Theta_k(n+1) \times E^k$. We shall compute $v_k(.,.,p)$ successively on $\mathbb{T}_p^2(k) \times \mathbb{R}^d$ (if it is not empty) and then on $\mathbb{T}_p^1(k) \times \mathbb{R}^d$, and we distinguish the two cases:

(i) $\mathbb{T}_p^2(k) = \emptyset$. This means $t_k + h \geq t_1 + mh$ or $t_k + h > T - mh$, and so $\mathbb{T}_p(k) = \mathbb{T}_p^1(k) = [t_k, t_1 + mh)$. We then compute $v_k(.,.,p)$ on $\mathbb{T}_p(k) \times \mathbb{R}^d$ as above for $k = m$:

$$\begin{aligned} v_k(t, x, p) = \mathbb{E}\Bigg[&\int_t^{t_1+mh} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t_1+mh}^{t,x,0}, e_1) \\ &+ v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\Bigg], \end{aligned}$$

where the r.h.s. is known from step n since either $p_- \in \Theta_{k-1}(n) \times E^{k-1}$ when $k > 1$, or $t_1 + mh \in \mathbb{T}^n(0)$ when $k - 1 = 0$.

(ii) $\mathbb{T}_p^2(k) \neq \emptyset$. This means $t_k + h < t_1 + mh$ and $t_k + h \leq T - mh$, so $\mathbb{T}_p^1(k) = [t_k, t_k + h) \cup ([t_k, t_k + h) \cap (T - mh, T))$, $\mathbb{T}_p^2(k) = [t_k + h, t_1 + mh) \cap [0, T - mh]$. For all $(t, x) \in \mathbb{T}_p^2(k) \times \mathbb{R}^d$, and $e \in E$, we have $p' = p \cup (t, e) \in \Theta_{k+1}(n + 1) \times E^{k+1}$, and $(t, x) \in \mathbb{T}_{p'}(k + 1) \times \mathbb{R}^d$. Hence, from the induction hypothesis at order $k + 1$, we know the value of the function:

$$F_{k,p}(t, x) = \sup_{e\in E} v_{k+1}(t, x, p \cup (t, e)), \quad (t, x) \in \mathbb{T}_p^2(k) \times \mathbb{R}^d.$$

We also know from step n the value of the function:

$$G_{k,p}(x) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-), \quad x \in \mathbb{R}^d.$$

Then, from the PDE (4.2) and the terminal condition (4.4) at k, we compute $v_k(.,.,p)$ on $\mathbb{T}_p^2(k) \times \mathbb{R}^d$ as the solution to an optimal stopping problem with obstacle $F_{k,p}$ and terminal condition $G_{k,p}$:

$$\begin{aligned} v_k(t, x, p) = \sup_{\tau\in\mathcal{T}_{t,t_1+mh}} & \mathbb{E}[F_{k,p}(\tau, X_\tau^{t,x,0}) 1_{\tau<t_1+mh} \\ & + G_{k,p}(X_{t_1+mh}^{t,x,0}) 1_{\tau=t_1+mh}], \quad (t, x) \in \mathbb{T}_p^2(k) \times \mathbb{R}^d. \end{aligned}$$

In particular, by continuity of $v_k(.,.,p)$ on $\mathbb{T}_p(k)$, we know the value of $\lim_{t\nearrow t_k+h} v_k(t, x, p) = v_k(t_k + h, p)$. We then compute $v_k(.,.,p)$ on $\mathbb{T}_p^1(k) \times \mathbb{R}^d$ from (4.1):

$$v_k(t, x, p) = \mathbb{E}\left[\int_t^{t_k+h} f(X_s^{t,x,0})\mathrm{d}s + v_k(t_k + h, X_{t_k+h}^{t,x,0}, p)\right].$$

We have then computed the value of $v_k(.,.,p)$ on $\mathbb{T}_p(k) \times \mathbb{R}^d$.

• From $k = 1 \to k = 0$. From the above item, we know the value of $v_1(.,.,p)$ on $\mathbb{T}_p(1) \times \mathbb{R}^d$ for all $p \in \Theta_1(n + 1) \times E$. Hence, we know the value of:

$$F_0(t, x) = \sup_{e\in E} v_1(t, x, (t, e)), \quad \forall(t, x) \in \mathbb{T}^{n+1}(0) \times \mathbb{R}^d.$$

From (5.2), we then compute v_0 on $\mathbb{T}^{n+1}(0) \times \mathbb{R}^d$ as an optimal stopping problem with obstacle F_1.

We have then calculated $v_k(.,.,p)$ on $\mathbb{T}_p(k) \times \mathbb{R}^d$ for all $p \in \Theta_k(n + 1) \times E^k$ and v_0 on $\mathbb{T}^{n+1}(0) \times \mathbb{R}^d$, and step $n + 1$ is stated. Finally, at step $n = N$, the computation of the value functions is completed since $\mathcal{D}_k(N) = \mathcal{D}_k$, $k = 0, \ldots, m$.

5.2. *Description of the optimal impulse control*

In view of the above dynamic programming relations, and the general theory of optimal stopping (see [8]), we can describe the structure of the optimal impulse control for $V_0 = v_0(0, X_0)$ in terms of the value functions. Let us define the following quantities:

► **Initialization**: $n = 0$

- *given an initial pending order number* $k = 0$, we define

$$\tilde{\tau}_1^{(0)} = \inf\left\{t \geq 0 : v_0(t, X_t^{\alpha^*}) = \sup_{e\in E} v_1(t, X_t^{\alpha^*}, (t, e))\right\} \wedge T,$$

$$\tilde{e}_1^{(0)} \in \arg\max_{e\in E} v_1(\tilde{\tau}_1^{(0)}, X_{\tilde{\tau}_1^{(0)}}^{\alpha^*}, (\tilde{\tau}_1^{(0)}, e)).$$

If $\tilde{\tau}_1^{(0)} + mh > T$, we stop the induction at $n = 0$, otherwise continue to the next item:

- *Pending orders number* $k \to k+1$ (this step is empty when $m = 1$) from $k = 1$:

$$\begin{aligned}\tilde{\tau}_{k+1}^{(0)} &= \inf\Big\{t \geq \tilde{\tau}_k^{(0)} + h : \\ &\qquad v_k(t, X_t^{\alpha^*}) = \sup_{e\in E} v_{k+1}(t, X_t^{\alpha^*}, (\tilde{\tau}_i^{(0)}, \tilde{e}_i^{(0)})_{1\leq i\leq k} \cup (t,e))\Big\} \wedge T,\\ \tilde{e}_{k+1}^{(0)} &\in \arg\max_{e\in E} v_{k+1}(\tilde{\tau}_{k+1}^{(0)}, X^{\alpha^*}_{\tilde{\tau}_{k+1}^{(0)}}, (\tilde{\tau}_i^{(n)}, \tilde{e}_i^{(0)})_{1\leq i\leq k} \cup (\tilde{\tau}_{k+1}^{(0)}, e)).\end{aligned}$$

As long as $\tilde{\tau}_k^{(0)} \leq \tilde{\tau}_1^{(0)} + mh$, increment $k \to k+1$: $\tilde{\tau}_k^{(0)} \to \tilde{\tau}_{k+1}^{(0)}$, until

$$k_0 = \sup\Big\{k : \tilde{\tau}_k^{(0)} \leq \tilde{\tau}_1^{(0)} + mh\Big\} \in \{1, \ldots, m\},$$

and increment the induction on n by the following step:

▶ $n \to n+1$:

- *given an initial pending orders number* $k = k_n - 1$, we define

$$\begin{aligned}\tilde{\tau}_{k_n}^{(n+1)} &= \inf\Big\{t \geq (\tilde{\tau}_1^{(n)} + mh) \vee (\tilde{\tau}_{k_n}^{(n)} + h) : \\ &\qquad v_{k_n-1}(t, X_t^{\alpha^*}, \tilde{p}_{n^-}) = \sup_{e\in E} v_{k_n}(t, X_t^{\alpha^*}, \tilde{p}_{n^-} \cup (t,e))\Big\} \wedge T,\\ \tilde{e}_{k_n}^{(n+1)} &\in \arg\max_{e\in E} v_{k_n}(\tilde{\tau}_1^{n+1}, X^{\alpha^*}_{\tilde{\tau}_1^{n+1}}, \tilde{p}_{n^-} \cup (\tilde{\tau}_{k_n}^{n+1}, e)),\end{aligned}$$

where we set $\tilde{p}_{n^-} = (\tilde{\tau}_i^{(n)}, \tilde{e}_i^{(n)})_{2\leq i\leq k_n}$. We denote $\tilde{\tau}_1^{(n+1)} = \tilde{\tau}_2^{(n)}$ if $k_n > 1$, and $\tilde{\tau}_1^{(n+1)} = \tilde{\tau}_{k_n}^{n+1}$ if $k_n = 1$. If $\tilde{\tau}_1^{(n+1)} + mh > T$, we stop the induction at $n+1$, otherwise continue to the next item:

- *Pending orders number* $k \to k+1$ (this step is empty when $m = 1$) from $k = k_n$:

$$\begin{aligned}\tilde{\tau}_{k+1}^{(n+1)} &= \inf\Big\{t \geq \tilde{\tau}_k^{(n+1)} + h : \\ &\qquad v_k(t, X_t^{\alpha^*}) = \sup_{e\in E} v_{k+1}(t, X_t^{\alpha^*}, \tilde{p}_{n^-} \cup (\tilde{\tau}_i^{(n+1)}, \tilde{e}_i^{(n+1)})_{k_n\leq i\leq k} \cup (t,e))\Big\} \wedge T\\ \tilde{e}_{k+1}^{(n+1)} &\in \arg\max_{e\in E} v_{k+1}(\tilde{\tau}_{k+1}^{(n+1)}, X^{\alpha^*}_{\tilde{\tau}_{k+1}^{(n+1)}}, \tilde{p}_{n^-} \cup (\tilde{\tau}_i^{(n+1)}, \tilde{e}_i^{(n+1)})_{k_n\leq i\leq k} \cup (\tilde{\tau}_{k+1}^{(n+1)}, e)).\end{aligned}$$

As long as $\tilde{\tau}_k^{(n+1)} \leq \tilde{\tau}_1^{(n+1)} + mh$, increment $k \to k+1$: $\tilde{\tau}_k^{(n+1)} \to \tilde{\tau}_{k+1}^{(n+1)}$, until

$$k_{n+1} = \sup\Big\{k : \tilde{\tau}_k^{(n+1)} \leq \tilde{\tau}_1^{(n+1)} + mh\Big\} \in \{1, \ldots, m\},$$

and continue the induction on n: $n \to n+1$ until $\tilde{\tau}_1^{(n+1)} + mh > T$.

The optimal impulse control is given by the finite sequence $\{(\tilde{\tau}_k^{(n)}, \tilde{e}_k^{(n)})_{k_{n-1}\leq k\leq k_n}, n = 0, \ldots, N\}$, where $N = \inf\{n \geq 0 : \tilde{\tau}_1^{(n)} + mh > T\}$, and we set by convention $k_{-1} = 1$.

6. Proof of viscosity properties

In this section, we prove the viscosity property stated in Proposition 4.2. We first state an auxiliary result. For any locally bounded function u on $\mathcal{D}_{k+1}$, $k = 0, \ldots, m-1$, we define the locally bounded function $\mathcal{H}u$ on $\mathcal{D}_k^2$ by $\mathcal{H}u(t,x,p) = \sup_{e\in E} u(t,x,p \cup (t,e))$.

Lemma 6.1. *Let u be a locally bounded function on $\mathcal{D}_{k+1}$, $k = 0, \ldots, m-1$. Then, $\mathcal{H}\overline{u}$ is upper-semicontinuous, and $\overline{\mathcal{H}u} \le \mathcal{H}\overline{u}$.*

Proof. Fix some $(t, x, p) \in \mathcal{D}_k^2$, and let $(t_n, x_n, p_n)_{n\ge1}$ be a sequence in $\mathcal{D}_k^2$ converging to (t, x, p) as n goes to infinity. Since $\overline{u}$ is upper-semicontinuous, and E is compact, there exists a sequence $(e_n)_n$ valued in E, such that

$$\mathcal{H}\overline{u}(t_n, x_n, p_n) = \overline{u}(t_n, x_n, p_n \cup (t_n, e_n)), \quad n \ge 1.$$

The sequence $(e_n)_n$ converges, up to a subsequence, to some $\hat{e} \in E$, and so

$$\begin{aligned}\mathcal{H}\overline{u}(t, x, p) &\ge \overline{u}(t, x, p \cup (t, \hat{e})) \ge \limsup_{n\to\infty} \overline{u}(t_n, x_n, p_n \cup (t_n, e_n)) \\ &= \limsup_{n\to\infty} \mathcal{H}\overline{u}(t_n, x_n, p_n),\end{aligned}$$

which shows that $\mathcal{H}\overline{u}$ is upper-semicontinuous.

On the other hand, fix some $(t, x, p) \in \mathcal{D}_k^{2,m}$, and let $(t_n, x_n, p_n)_{n\ge1}$ be a sequence in $\mathcal{D}_k^2$ converging to (t, x, p) s.t. $\mathcal{H}u(t_n, x_n, p_n)$ converges to $\overline{\mathcal{H}u}(t, x, p)$. Then, we have

$$\overline{\mathcal{H}u}(t, x, p) = \lim_{n\to\infty} \mathcal{H}u(t_n, x_n, p_n) \le \limsup_{n\to\infty} \mathcal{H}\overline{u}(t_n, x_n, p_n) \le \mathcal{H}\overline{u}(t, x, p),$$

which shows that $\overline{\mathcal{H}u} \le \mathcal{H}\overline{u}$. □

Now, we prove the sub and supersolution property of the family v_k, $k = 0, \ldots, m$. There is no difficulty on the domain $\mathcal{D}_k^1$ since locally no impulse control is possible. Hence, in this case, the viscosity properties can be derived as for an uncontrolled state process, and the proof is standard from the dynamic programming principle (3.7), see e.g. [13]. Notice that since the domain $\mathbb{T}_p^1(k)$ is open in $\mathbb{T}_p(k)$, we have no problem at the boundary. Indeed, this set is open at $(t_k + h) \wedge (t_1 + mh)$ and eventually $T - mh$, which is the usual situation, and the closedness at t_k and T does not introduce difficulties, as the value function is not defined before t_k and after T. Hence, when taking approximations of the upper- and lower-semicontinuous envelopes of v_k, we only need to consider points of the domain such that $t \ge t_k$, where the dynamic programming relation (3.7) holds. The proof of the viscosity property of the value functions v_k to (4.2) on $\mathcal{D}_k^2$ is more subtle. Indeed, in addition to the specific form of Eq. (4.2), we have to carefully address the discontinuity of the PDE system (4.1) and (4.2) on the boundaries $t_k + h$ and eventually $T - mh$ of $\mathbb{T}_p^2(k)$. In what follows, we focus on the domain $\mathcal{D}_k^2$, $k = 0, \ldots, m-1$.

Proof of the supersolution property on $\mathcal{D}_k^2$.

We first prove that for $k = 0, \ldots, m-1$, $(t_0, x_0, p_0) \in \mathcal{D}_k^2$:

$$\underline{v_k}(t_0, x_0, p_0) \ge \sup_{e\in E} \underline{v_{k+1}}(t_0, x_0, p_0 \cup (t_0, e)). \tag{6.1}$$

By definition of $\underline{v_k}$, there exists a sequence $(t_n, x_n, p_n)_{n\ge1} \in \mathcal{D}_k^m$ such that:

$$v_k(t_n, x_n, p_n) \to \underline{v_k}(t_0, x_0, p_0) \quad \text{with } (t_n, x_n, p_n) \to (t_0, x_0, p_0). \tag{6.2}$$

We set $p_0 = (t_i^0, e_i^0)_{1\le i\le k}$, $p_n = (t_i^n, e_i^n)_{1\le i\le k}$, and we distinguish the three following cases:

• If $t_k^0 + h < t_0 < T - mh$, then, for n sufficiently large, we have $t_k^n + h \le t_n \le T - mh$, i.e. $p_n \in P_{t_n}^2(k)$. Hence, from the dynamic programming principle by making an immediate impulse control, i.e. by applying (3.8) to $v_k(t_n, x_n, p_n)$ with $\theta = \tau = t_n$, and $e \in E$, we have

$$v_k(t_n, x_n, p_n) \ge v_{k+1}(t_n, x_n, p_n \cup (t_n, e)) \ge \underline{v_{k+1}}(t_n, x_n, p_n \cup (t_n, e)).$$

By sending n to infinity with (6.2), and since $\underline{v_{k+1}}$ is lower-semicontinuous, we obtain the required relation (6.1) from the arbitrariness of e in E.

• if $t_0 = t_k^0 + h \neq T - mh$, we apply the dynamic programming principle by making an impulse control as soon as possible. This means that in relation (3.5) for $v_k(t_n, x_n, p_n)$, we choose $\alpha = (\tau_i, \xi_i)_{i\geq 1} \in \mathcal{A}_{t_n,p_n}$, $\theta = \tau_{k+1} = \theta_n := t_n \vee (t_k^n + h)$, $\xi_{k+1} = e \in E$, so that:

$$v_k(t_n, x_n, p_n) \geq \mathbb{E}\Bigg[\int_{t_n}^{\theta_n} f(X_s^n)\mathrm{d}s + \sum_{t_n<\tau_i+mh\leq\theta_n} c(X^n_{(\tau_i+mh)^-}, \xi_i) + \underline{v_{k+1}}(\theta_n, X^n_{\theta_n}, p_n \cup (\theta_n, e))\Bigg].$$

Here $X^n := X^{t_n,x_n,0}$. Since $t_n, \theta_n \to t_0$, $p_n \to p_0$, $X^n_{\theta_n} \to x_0$ a.s., as n goes to infinity, and from estimate (2.8) and the linear growth condition on f, c, $\underline{v_{k+1}}$, we can use the dominated convergence theorem to obtain:

$$\underline{v_k}(t_0, x_0, p_0) \geq \underline{v_{k+1}}(t_0, x_0, p_0 \cup (t_0, e)),$$

which implies (6.1) from the arbitrariness of $e \in E$.

• if $t_0 = T - mh$, we show from condition (2.7) that it is not optimal to decide an impulse intervention. First, notice from the definition of the value function and from the constraints on the impulse controls that, for all $e \in E$:

$$v_{k+1}(t_0, x_0, p_0 \cup (t_0, e)) = \mathbb{E}\Bigg[\int_{t_0}^{T} f(X_s^{t_0,x_0,p_0})\mathrm{d}s + g\left(\Gamma(X^{t_0,x_0,p_0}_{T^-}, e)\right) + \sum_{i=0}^{k} c(X^{t_0,x_0,p_0}_{(t_i^0+mh)^-}, e_i^0) + c(X^{t_0,x_0,p_0}_{T^-}, e)\Bigg]. \tag{6.3}$$

Moreover, by definition of v_k, and by choosing not to decide an impulse intervention, we get for all n:

$$v_k(t_n, x_n, p_n) \geq \mathbb{E}\left[\int_{t_n}^{T} f(X_s^{t_n,x_n,p_n})\mathrm{d}s + g\left(X^{t_n,x_n,p_n}_{T^-}\right) + \sum_{i=0}^{k} c(X^{t_n,x_n,p_n}_{(t_i^n+mh)^-}, e_i^n)\right].$$

Hence, by the continuity and the linear growth conditions of f, g, Γ, c together with the dominated convergence theorem, we get by sending n to infinity into the previous inequality:

$$\underline{v_k}(t_0, x_0, p_0) \geq \mathbb{E}\left[\int_{t_0}^{T} f(X_s^{t_0,x_0,p_0})\mathrm{d}s + g\left(X^{t_0,x_0,p_0}_{T^-}\right) + \sum_{i=0,..,k} c(X^{t_0,x_0,p_0}_{(t_{i,0}+mh)^-}, e_{i,0})\right].$$

Finally, by using Assumption (2.7) and equality (6.3), we get:

$$\underline{v_k}(t_0, x_0, p_0) \geq v_{k+1}(t_0, x_0, p_0 \cup (t_0, e)) \geq \underline{v_{k+1}}(t_0, x_0, p_0 \cup (t_0, e)),$$

which proves the required inequality from the arbitrariness of e in E.

Finally, in order to complete the viscosity supersolution property of v_k to (4.2) on $\mathcal{D}_k^2$, it remains to show that v_k is a supersolution to:

$$-\frac{\partial v_k}{\partial t}(t, x, p) - \mathcal{L}v_k(t, x, p) - f(x) \geq 0,$$

on $\mathcal{D}_k^2$. This proof is standard by using the dynamic programming relation (3.8) with $\tau = \infty$ and Itô's formula, see [13] for the details. $\square$

Proof of the subsolution property on $\mathcal{D}_k^2$.

We follow arguments in [10]. Let $(t_0, x_0, p_0) \in \mathcal{D}_k^{2,m}$ and $\varphi \in C^{1,2}(\mathcal{D}_k^2)$ such that $\overline{v_k}(t_0, x_0, p_0) = \varphi(t_0, x_0, p_0)$ and $\varphi \geq \overline{v_k}$ on $\mathcal{D}_k^2$. If $\overline{v_k}(t_0, x_0, p_0) \leq \mathcal{H}\overline{v_{k+1}}(t_0, x_0, p_0)$, then the subsolution inequality holds trivially. Now, if $\overline{v_k}(t_0, x_0, p_0) > \mathcal{H}\overline{v_{k+1}}(t_0, x_0, p_0)$, we argue by contradiction by assuming on the contrary that

$$\eta := -\frac{\partial \varphi}{\partial t}(t_0, x_0, p_0) - \mathcal{L}\varphi(t_0, x_0, p_0) - f(x_0) > 0.$$

We set $p_0 = (t_i^0, e_i^0)_{1 \leq i \leq k}$. By continuity of φ and its derivatives, there exists some $\delta > 0$ with $t_0 + \delta < (t_1^0 + mh) \wedge T$ such that:

$$-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi - f > \frac{\eta}{2}, \quad \text{on } ((t_0 - \delta, t_0 + \delta) \times B(x_0, \delta) \times B(p_0, \delta)) \cap \mathcal{D}_k^{2,m}. \tag{6.4}$$

From the definition of $\overline{v_k}$, there exists a sequence $(t_n, x_n, p_n)_{n \geq 1} \in ((t_0 - \delta, t_0 + \delta) \times B(x_0, \delta) \times B(p_0, \delta)) \cap \mathcal{D}_k^2$ such that $(t_n, x_n, p_n) \to (t_0, x_0, p_0)$ and $v_k(t_n, x_n, p_n) \to \overline{v_k}(t_0, x_0, p_0)$ as $n \to \infty$. By continuity of φ we also have that $\gamma_n := v_k(t_n, x_n, p_n) - \varphi(t_n, x_n, p_n)$ converges to 0 as $n \to \infty$. We set $p_n = (t_i^n, e_i^n)_{1 \leq i \leq k}$. From the dynamic programming principle (3.8), for each $n \geq 1$, there exists a control $(\tau^n, \xi^n) \in \mathcal{I}_{t_n}$ such that

$$\begin{aligned} v_k(t_n, x_n, p_n) - \frac{\eta}{4}\delta_n \leq \mathbb{E}\Bigg[\int_{t_n}^{\theta_n} & f(X_s^n)\mathrm{d}s + v_k(\theta_n, X_{\theta_n}^n, p_n) 1_{\theta_n < \tau_n} \\ & + v_{k+1}(\theta_n, X_{\theta_n}, p_n \cup (\tau_n, \xi_n)) 1_{\tau_n \leq \theta_n}\Bigg]. \end{aligned} \tag{6.5}$$

Here $X^n := X^{t_n, x_n, 0}$, we choose $\theta_n = \vartheta_n \wedge (t_n + \delta_n)$, with $\vartheta_n = \inf\{s \geq t_n : X_s^n \notin B(x_n, \frac{\delta}{2})\}$, and $(\delta_n)_n$ is a strictly positive sequence such that

$$\delta_n \to 0, \qquad \frac{\gamma_n}{\delta_n} \to 0, \quad \text{as } n \to \infty.$$

On the other hand, from Lemma 6.1, we have

$$\overline{\mathcal{H}v_{k+1}}(t_0, x_0, p_0) \leq \mathcal{H}\overline{v_{k+1}}(t_0, x_0, p_0) < \overline{v_k}(t_0, x_0, p_0) \leq \varphi(t_0, x_0, p_0).$$

Hence, since $\overline{\mathcal{H}v_{k+1}}$ is u.s.c. and φ is continuous, the inequality $\mathcal{H}v_{k+1} \leq \varphi$ holds in a neighborhood of (t_0, x_0, p_0), and so for sufficiently large n, we get:

$$v_{k+1}(\theta_n, X_{\theta_n}^n, p_n \cup (\tau_n, \xi_n)) 1_{\tau_n \leq \theta_n} \leq \varphi(\theta_n, X_{\theta_n}^n, p_n) 1_{\tau_n \leq \theta_n} \quad a.s.$$

Together with (6.5), this yields:

$$\varphi(t_n, x_n, p_n) + \gamma_n - \frac{\eta}{4}\delta_n \leq \mathbb{E}\left[\int_{t_n}^{\theta_n} f(X_s^n)\mathrm{d}s + \varphi(\theta_n, X_{\theta_n}^n, p_n)\right].$$

By applying Itô's formula to $\varphi(s, X_s^n, p_n)$ between $s = t_n$ and $s = \theta_n$, and dividing by δ_n, we then get:

$$\frac{\gamma_n}{\delta_n} - \frac{\eta}{4} \leq \frac{1}{\delta_n}\mathbb{E}\left[\int_{t_n}^{\theta_n} \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi + f\right)(s, X_s^n, p_n)\mathrm{d}s\right] \leq -\frac{\eta}{2}\mathbb{E}\left[\frac{\theta_n - t_n}{\delta_n}\right], \tag{6.6}$$

from (6.4). Now, from the growth linear condition on b, σ, Burkholder–Davis–Gundy inequality and Gronwall's lemma, we have the standard estimate: $\mathbb{E}[\sup_{s\in[t_n,t_n+\delta_n]}|X_s^n - x_n|^2] \to 0$, so that by Chebichev inequality, $\mathbb{P}[\vartheta_n \le t_n + \delta_n] \to 0$, as n goes to infinity, and therefore by definition of θ_n:

$$1 \ge \mathbb{E}\left[\frac{\theta_n - t_n}{\delta_n}\right] \ge \mathbb{P}[\vartheta_n > t_n + \delta_n] \to 1, \quad \text{as } n \to \infty.$$

By sending n to infinity into (6.6), we obtain the required contradiction: $-\frac{\eta}{4} \le -\frac{\eta}{2}$. $\square$

7. Proofs of comparison principles and uniqueness results

7.1. Sequential comparison results

In this paragraph, we prove sequential comparison results. We consider the sets $\Theta_k(n)$, $\mathbb{T}^n(0)$, $\mathcal{D}_k(n)$, and $\mathcal{D}_k^i(n)$, introduced in Section 5 for $n = m, \dots N$, and $k = 0, \dots, m(n)$, and we define sequential viscosity solutions as follows.

Definition 7.1. Let $n \in \{m+1, \dots, N\}$. We say that a family of locally bounded functions w_k on $\mathcal{D}_k(n)$, $k = 0, \dots, m(n)$, is a viscosity supersolution (resp. subsolution) of (4.1) and (4.2) at step n if:

(i) for all $k = 0, \dots, m(n)$, $(t_0, x_0, p_0) \in \mathcal{D}_k^1(n)$, and $\varphi \in C^{1,2}(\mathcal{D}_k^1(n))$, which realizes a local minimum of $\underline{w_k} - \varphi$ (resp. maximum of $\overline{w_k} - \varphi$), we have

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0, p_0) - \mathcal{L}\varphi(t_0, x_0, p_0) - f(x_0) \ge 0 \quad (\text{resp.} \le 0).$$

(ii) for all $k = 0, \dots, m(n) - 1$, $(t_0, x_0, p_0) \in \mathcal{D}_k^2(n)$, and $\varphi \in C^{1,2}(\mathcal{D}_k^2(n))$, which realizes a local minimum of $\underline{w_k} - \varphi$ (resp. maximum of $\overline{w_k} - \varphi$), we have

$$\min\left\{-\frac{\partial \varphi}{\partial t}(t_0, x_0, p_0) - \mathcal{L}\varphi(t_0, x_0, p_0) - f(x_0), \right.$$
$$\left. \underline{w_k}(t_0, x_0, p_0) - \sup_{e\in E} \underline{w_{k+1}}(t_0, x_0, p_0 \cup (t_0, e))\right\} \ge 0$$

(resp.

$$\min\left\{-\frac{\partial \varphi}{\partial t}(t_0, x_0, p_0) - \mathcal{L}\varphi(t_0, x_0, p_0) - f(x_0), \right.$$
$$\left. \overline{w_k}(t_0, x_0, p_0) - \sup_{e\in E} \overline{w_{k+1}}(t_0, x_0, p_0 \cup (t_0, e))\right\} \le 0).$$

We say that a family of locally bounded functions w_k on $\mathcal{D}_k(n)$, $k = 0, \dots, m(n)$, is a viscosity solution of (4.1) and (4.2) at step n if it is a viscosity supersolution and subsolution of (4.1) and (4.2) at step n.

We then prove the following comparison principle at step n.

Proposition 7.2. *Let $n \in \{m+1, \dots, N\}$. Let u_k (resp. w_k), $k = 0, \dots, m(n)$, be a family of viscosity subsolution (resp. supersolution) of (4.1) and (4.2) at step n satisfying growth* *condition (2.9). Suppose also that w_k satisfies (4.3). If u_k and w_k are such that for all $x \in \mathbb{R}^d$*

$$\overline{u_k}(t_1 + mh, x, p) \leq \underline{w_k}(t_1 + mh, x, p), \quad p = (t_i, e_i)_{1 \leq i \leq k} \in \Theta_k(n) \times E^k, k \geq 1,$$
$$\overline{u_0}(T, x) \leq \underline{w_0}(T, x).$$

Then, $\overline{u_k} \leq \underline{w_k}$ on $\mathcal{D}_k(n)$, $k = 0, \ldots, m(n)$.

Remark 7.3. We recall some basic definitions and properties in viscosity solutions theory, which shall be used in the proof of the above proposition. Consider the general PDE

$$F\left(t, x, w, \frac{\partial w}{\partial t}, D_x w, D_x^2 w\right) = 0 \quad \text{on } [t_0, t_1) \times \mathcal{O}, \tag{7.1}$$

where $t_0 < t_1$, and $\mathcal{O}$ is an open set in $\mathbb{R}^d$. There is an equivalent definition of viscosity solutions to (7.1) in terms of semijets $\bar{J}^{2,+}w(t,x)$ and $\bar{J}^{2,-}w(t,x)$ associated respectively to an upper-semicontinuous (u.s.c.) and lower-semicontinuous (l.s.c.) function w (see [7] or [9] for the definition of semijets): an u.s.c. (resp. l.s.c.) function w is a viscosity subsolution (resp. supersolution) to (7.1) if and only if for all $(t, x) \in [t_0, t_1) \times \mathcal{O}$,

$$F(t, x, w(t, x), r, q, A) \leq (\text{resp. } \geq) 0, \quad \forall (r, q, A) \in \bar{J}^{2,+}w(t, x) (\text{resp. } \bar{J}^{2,-}w(t, x)).$$

For $\eta > 0$, we say that w^η is a viscosity η-strict supersolution to (7.1), if w^η is a viscosity supersolution to

$$F\left(t, x, w^\eta, \frac{\partial w^\eta}{\partial t}, D_x w^\eta, D_x^2 w^\eta\right) \geq \eta, \quad \text{on } [t_0, t_1) \times \mathcal{O}$$

in the sense that it is a viscosity supersolution to $F(t, x, w^\eta, \frac{\partial w^\eta}{\partial t}, D_x w^\eta, D_x^2 w^\eta) - \eta = 0$, on $[t_0, t_1) \times \mathcal{O}$.

As usual when dealing with variational inequalities, we begin the proof of the comparison principle by showing the existence of viscosity η-strict supersolutions for Eqs. (4.1) and (4.2).

Lemma 7.4. *Let $n \in \{m+1, \ldots, N\}$. Let w_k, $k = 0, \ldots, m(n)$, be a family of viscosity supersolutions of (4.1) and (4.2) satisfying (4.3). Then, for any $\eta > 0$, there exists a family of viscosity η-strict supersolutions w_k^η of (4.1) and (4.2) such that for $k = 0, \ldots, m(n)$:*

$$w_k(t, x, p) + \eta C_1 |x|^2 \leq w_k^\eta(t, x, p) \leq w_k(t, x, p) + \eta C_2 (1 + |x|^2), \quad (t, x, p) \in \mathcal{D}_k, \tag{7.2}$$

for some positive constants C_1, C_2 independent on η. Moreover, for $k = 0, \ldots, m(n) - 1$, $(t, x, p) \in \mathcal{D}_k(n)$, $p = (t_i, e_i)_{1 \leq i \leq k}$ with $t = t_k + h$, we have:

$$\underline{w_k^\eta}(t, x, p) \geq \sup_{e \in E} \underline{w_{k+1}^\eta}(t, x, p \cup (t, e)) + \eta. \tag{7.3}$$

Proof. For $\eta > 0$, consider the functions:

$$w_k^\eta(t, x, p) = w_k(t, x, p) + \eta \phi_{1,k}(t) + \eta \phi_2(t, x), \phi_{1,k}(t) = [(T - t) + (m - k)],$$
$$\phi_2(t, x) = \frac{1}{2} e^{L(T-t)} \left(1 + |x|^2\right),$$

with L a positive constant to be determined later. It is clear that w_k^η satisfies (7.2) with $C_1 = 1/2$ and $C_2 = T + m + e^{LT}/2$. Moreover, we easily show that $w_k + \eta\phi_{1,k}^\eta$ is a viscosity supersolution to

$$- \frac{\partial(w_k + \eta\phi_{1,k})}{\partial t} - \mathcal{L}(w_k + \eta\phi_{1,k}) - f \geq \eta. \tag{7.4}$$

This is derived from the fact that $-\frac{\partial\phi_{1,k}}{\partial t} - \mathcal{L}\phi_{1,k} = 1$, and w_k is a viscosity supersolution to $-\frac{\partial w_k}{\partial t} - \mathcal{L}w_k - f \geq 0$. We now show that ϕ_2 is a supersolution to

$$- \frac{\partial\phi_2}{\partial t} - \mathcal{L}\phi_2 \geq 0. \tag{7.5}$$

This is carried out by calculating this quantity explicitly. Indeed, we have

$$\frac{\partial\phi_2}{\partial t}(t,x) = -\frac{L}{2}e^{L(T-t)}(1+|x|^2), \qquad \mathcal{L}\phi_2(t,x) = e^{L(T-t)}\left(b(x).x + \mathrm{tr}\left(\sigma\sigma'(x)\right)\right).$$

Since b and σ are of linear growth, we thus obtain:

$$-\frac{\partial\phi_2}{\partial t}(t,x) - \mathcal{L}\phi_2(t,x) \geq e^{L(T-t)}\left[\frac{L}{2}(1+|x|^2) - C(1+|x|+|x|^2)\right],$$

for some constant C independent of t, x. Therefore, by taking L sufficiently large, we get the required inequality (7.5), which shows together with (7.4) that w_k^η is a viscosity supersolution to

$$- \frac{\partial w_k^\eta}{\partial t} - \mathcal{L}w_k^\eta - f \geq \eta. \tag{7.6}$$

Moreover, since $\underline{w_k}(t,x,p) - \sup_{e\in E}\underline{w_{k+1}}(t,x,p\cup(t,e)) \geq 0$, we immediately get

$$\begin{aligned}
&\underline{w_k^\eta}(t,x,p) - \sup_{e\in E}\underline{w_{k+1}^\eta}(t,x,p\cup(t,e)) \\
&\quad = \underline{w_k}(t,x,p) + \eta\phi_{1,k}(t) - \sup_{e\in E}\underline{w_{k+1}}(t,x,p\cup(t,e)) - \eta\phi_{1,k+1}(t) \\
&\quad \geq \eta\phi_{1,k}(t) - \eta\phi_{1,k+1}(t) \geq \eta.
\end{aligned}$$

Together with (7.6), this proves the required viscosity η-strict supersolution property for w_k^η to (4.1) and (4.2). □

The main step in the proof of Proposition 7.2 consists in the comparison principle for η-strict supersolutions. Notice from (7.2) that once w_k satisfies a linear growth condition, then w_k^η satisfies the quadratic growth lower-bound condition:

$$\eta C_1 |x|^2 - C_2 \leq w_k^\eta(t,x,p), \quad (t,x,p) \in \mathcal{D}_k, \tag{7.7}$$

for some positive constants C_1, C_2.

Lemma 7.5. *Let $n \in \{m+1,\ldots,N\}$ and $\eta > 0$. Let u_k (resp. w_k), $k = 0,\ldots,(n-m)\wedge m$, be a family of viscosity subsolution (resp. η-strict supersolution) of (4.1) and (4.2) at step n, with u_k satisfying the linear growth condition (2.9) and w_k satisfying the quadratic growth condition (7.7). Suppose that for all $x \in \mathbb{R}^d$,*

$$\overline{u_k}(t_1+mh,x,p) \leq \underline{w_k}(t_1+mh,x,p), \quad p = (t_i,e_i)_{1\leq i\leq k} \in \Theta_k(n)\times E^k, k \geq 1, \tag{7.8}$$

$$\overline{u_0}(T,x) \leq \underline{w_0}(T,x). \tag{7.9}$$

$$\underline{w_k}(t_k+h,x,\pi) \geq \sup_{e\in E} \underline{w_{k+1}}(t_k+h,x,p\cup(t_k+h,e))+\eta,$$
$$p=(t_i,e_i)_{1\leq i\leq k}\in \Theta_k(n)\times E^k, \quad k\leq m-1. \tag{7.10}$$
$$\underline{w_k}(T-mh,x,\pi) \geq \sup_{e\in E} \underline{w_{k+1}}(T-mh,x,p\cup(T-mh,e))+\eta,$$
$$\text{for all } (T-mh,x,p)\in \mathcal{D}_k(n). \tag{7.11}$$

Then, $\overline{u_k} \leq \underline{w_k}$ *on* $\mathcal{D}_k(n)$, $k=0,\ldots,(n-m)\wedge m$.

Proof. From the linear growth of u_k, and from the quadratic growth lower-bound of w_k, we have

$$\overline{u_k}(t,x,p)-\underline{w_k}(t,x,p)\leq C_1(1+|x|)-C_2|x|^2, \quad k=0,\ldots,m, (t,x,p)\in\mathcal{D}_k(n),$$

for some positive constants C_1, C_2. Thus, for all k, the supremum of the u.s.c function $\overline{u_k}-\underline{w_k}$ is attained on a compact set that only depends on C_1 and C_2. Hence, one can find $k_0\in\{0,\ldots,(n-m)\wedge m\}$, $(t_0,x_0,p_0)\in\mathcal{D}_{k_0}(n)$ such that:

$$M := \sup_{\substack{k\in\{0,\ldots,m\}\\(t,x,p)\in\mathcal{D}_k(n)}} \left[\overline{u_k}(t,x,p)-\underline{w_k}(t,x,p)\right] = \overline{u_{k_0}}(t_0,x_0,p_0)-\underline{w_{k_0}}(t_0,x_0,p_0), \tag{7.12}$$

and we have to show that $M\leq 0$. We set $p_0=(t_i^0,e_i^0)_{1\leq i\leq k_0}$, and we distinguish the six possible cases concerning (k_0,t_0,x_0,p_0):

(1) $k_0\neq 0$, $t_0=t_1^0+mh$,
(2) $k_0=0$, $t_0=T$,
(3) $k_0\neq 0$, $t_0\in\mathbb{T}^1_{p_0}(k_0)$
(4) $k_0=0$, $t_0\in[0,T-mh)$ or $k_0\in\{1,\ldots,m-1\}$, $t_0\in\mathbb{T}^2_{p_0}(k_0)$, $t_0\neq t^0_{k_0}+h$, $t_0\neq T-mh$
(5) $k_0\in\{1,\ldots,m-1\}$, $t_0=t^0_{k_0}+h$,
(6) $k_0\in\{1,\ldots,m-1\}$, $t_0=T-mh$,

► *Cases* 1 *and* 2: these two cases imply directly from (7.8) (resp. (7.9)) that $M\leq 0$.

► *Cases* 3 *and* 4: we focus only on case 4, as case 3 involves similar (and simpler) arguments. We follow general viscosity solution technique based on the Ishii technique and work towards a contradiction. To this end, let us consider the following function:

$$\Phi_\varepsilon(t,t',x,x',p,p')=\overline{u_{k_0}}(t,x,p)-\underline{w_{k_0}}(t',x',p')-\psi_\varepsilon(t,t',x,x',p,p'),$$

with

$$\psi_\varepsilon(t,t',x,x',p,p')=\frac{1}{2}\left[|t-t_0|^2+|p-p_0|^2\right]+\frac{1}{4}|x-x_0|^4 + \frac{1}{2\varepsilon}\left[|t-t'|^2+|x-x'|^2+|p-p'|^2\right].$$

By the positiveness of the function ψ_ε, we notice that (t_0,x_0,p_0) is a strict maximizer of $(t,x,p)\to\Phi_\varepsilon(t,t,x,x,p,p)$. Hence, by Proposition 3.7 in [7], there exists a sequence of maximizers $(t_\varepsilon,t'_\varepsilon,x_\varepsilon,x'_\varepsilon,p_\varepsilon,p'_\varepsilon)$ of Φ_ε such that:

$$(t_\varepsilon,t'_\varepsilon,x_\varepsilon,x'_\varepsilon,p_\varepsilon,p'_\varepsilon)\to(t_0,t_0,x_0,x_0,p_0,p_0), \tag{7.13}$$
$$\overline{u_{k_0}}(t_\varepsilon,x_\varepsilon,p_\varepsilon)-\underline{w_{k_0}}(t'_\varepsilon,x'_\varepsilon,p'_\varepsilon)\to\overline{u_{k_0}}(t_0,x_0,p_0)-\underline{w_{k_0}}(t_0,x_0,p_0), \tag{7.14}$$
$$\frac{1}{\varepsilon}\left[|t_\varepsilon-t'_\varepsilon|^2+|x_\varepsilon-x'_\varepsilon|^2+|p_\varepsilon-p'_\varepsilon|^2\right]\to 0 \quad \text{as } \varepsilon\to 0. \tag{7.15}$$

By applying Theorem 3.2 in [7] to the sequence of maximizers $(t_\varepsilon, t'_\varepsilon, x_\varepsilon, x'_\varepsilon, p_\varepsilon, p'_\varepsilon)$ of Φ_ε, we get the existence of two symmetric matrices A_ε, A'_ε such that:

$$(r_\varepsilon, q_\varepsilon, A_\varepsilon) \in \overline{J}^{2,+}\overline{u_{k_0}}(t_\varepsilon, x_\varepsilon, p_\varepsilon), \qquad \left(r'_\varepsilon, q'_\varepsilon, A'_\varepsilon\right) \in \overline{J}^{2,-}\underline{w_{k_0}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon), \tag{7.16}$$

where

$$r_\varepsilon = \frac{\partial \psi_\varepsilon}{\partial t}(t_\varepsilon, t'_\varepsilon, x_\varepsilon, x'_\varepsilon, p_\varepsilon, p'_\varepsilon) = \frac{1}{\varepsilon}(t_\varepsilon - t'_\varepsilon) + (t_\varepsilon - t_0), \tag{7.17}$$

$$r'_\varepsilon = -\frac{\partial \psi_\varepsilon}{\partial t'}(t_\varepsilon, t'_\varepsilon, x_\varepsilon, x'_\varepsilon, p_\varepsilon, p'_\varepsilon) = \frac{1}{\varepsilon}(t_\varepsilon - t'_\varepsilon) \tag{7.18}$$

$$q_\varepsilon = \frac{\partial \psi_\varepsilon}{\partial x}(t_\varepsilon, t'_\varepsilon, x_\varepsilon, x'_\varepsilon, p_\varepsilon, p'_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon) + |x_\varepsilon - x_0|^2 (x_\varepsilon - x_0), \tag{7.19}$$

$$q'_\varepsilon = -\frac{\partial \psi_\varepsilon}{\partial x'}(t_\varepsilon, t'_\varepsilon, x_\varepsilon, x'_\varepsilon, p_\varepsilon, p'_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - x'_\varepsilon), \tag{7.20}$$

and

$$\begin{pmatrix} A_\varepsilon & 0 \\ 0 & -A'_\varepsilon \end{pmatrix} \le \begin{pmatrix} \frac{3}{\varepsilon} I_d - Q(x_\varepsilon - x_0) & -\frac{3}{\varepsilon} I_d \\ -\frac{3}{\varepsilon} I_d & \frac{3}{\varepsilon} I_d \end{pmatrix}, \tag{7.21}$$

with $Q(x) = 2x \otimes x + |x|^2 I_d$, I_d the identity matrix of dimension $d \times d$, and for $x = (x_i)_{1 \le i \le d} \in \mathbb{R}^d$, $x \otimes x$ is the tensorial product defined by $x \otimes x = (x_i x_j)_{i,j \in \{1..d\}^2}$. Here, to alleviate notations, and since there is no derivatives with respect to the variable p in the PDE, the semijets are defined with respect to the variables (t, x), and we omitted the terms corresponding to the derivatives of ψ_ε with respect to p. We set $p_\varepsilon = (t_i^\varepsilon, e_i^\varepsilon)_{1 \le i \le k_0}$, and $p'_\varepsilon = (t_i'^{\varepsilon}, e_i'^{\varepsilon})_{1 \le i \le k_0}$. From (7.13), we deduce that for ε small enough, $t_\varepsilon \in \mathbb{T}^2_{p_0}(k_0)$ and $t_\varepsilon \ne t^\varepsilon_{k_0} + h$. From (7.16), and the formulation of viscosity subsolution of u_{k_0} to (4.2) and η-strict viscosity supersolution of w_{k_0} to (4.2) by means of semijets, we have for all ε small enough:

$$\min\Big\{-r_\varepsilon - b(x_\varepsilon)q_\varepsilon - \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(x_\varepsilon)A_\varepsilon\right) - f(x_\varepsilon),$$

$$\overline{u_{k_0}}(t_\varepsilon, x_\varepsilon, p_\varepsilon) - \sup_{e \in E} \overline{u_{k_0+1}}(t_\varepsilon, x_\varepsilon, p_\varepsilon \cup (t_\varepsilon, e))\Big\} \le 0, \tag{7.22}$$

$$\min\Big\{-r'_\varepsilon - b(x'_\varepsilon)q'_\varepsilon - \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(x'_\varepsilon)A'_\varepsilon\right) - f(x'_\varepsilon),$$

$$\underline{w_{k_0}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon) - \sup_{e \in E} \underline{w_{k_0+1}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon \cup (t'_\varepsilon, e))\Big\} \ge \eta. \tag{7.23}$$

We then distinguish the following two possibilities in (7.22):

- (i) for all ε small enough,

$$\overline{u_{k_0}}(t_\varepsilon, x_\varepsilon, p_\varepsilon) - \sup_{e \in E} \overline{u_{k_0+1}}(t_\varepsilon, x_\varepsilon, p_\varepsilon \cup (t_\varepsilon, e)) \le 0.$$

Then, for all ε small enough, there exists $e_\varepsilon \in E$ such that:

$$\overline{u_{k_0}}(t_\varepsilon, x_\varepsilon, p_\varepsilon) \le \overline{u_{k_0+1}}(t_\varepsilon, x_\varepsilon, p_\varepsilon \cup (t_\varepsilon, e_\varepsilon)) + \frac{\eta}{2}.$$

Moreover, by (7.23), we have

$$\underline{w_{k_0}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon) \geq \underline{w_{k_0+1}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon \cup (t'_\varepsilon, e_\varepsilon)) + \eta.$$

Combining the above two inequalities, we deduce that for all ε small enough,

$$\begin{aligned}
&\overline{u_{k_0}}(t_\varepsilon, x_\varepsilon, p_\varepsilon) - \underline{w_{k_0}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon)\\
&\quad \leq \overline{u_{k_0+1}}(t_\varepsilon, x_\varepsilon, p_\varepsilon \cup (t_\varepsilon, e_\varepsilon)) - \underline{w_{k_0+1}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon \cup (t'_\varepsilon, e_\varepsilon)) - \frac{\eta}{2}.
\end{aligned}$$

Since E is compact, there exists some $e \in E$ s.t. $e_\varepsilon \to e$ up to a subsequence. From (7.13) and (7.14), and since $\overline{u_{k_0}}$, $-\underline{w_{k_0}}$ are u.s.c., we obtain by sending ε to zero:

$$\begin{aligned}
&\overline{u_{k_0}}(t_0, x_0, p_0) - \underline{w_{k_0}}(t_0, x_0, p_0)\\
&\quad \leq \overline{u_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)) - \underline{w_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)) - \frac{\eta}{2},
\end{aligned}$$

which contradicts (7.12).

• (ii) for all ε small enough,

$$-r_\varepsilon - b(x_\varepsilon)q_\varepsilon - \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(x_\varepsilon)A_\varepsilon\right) - f(x_\varepsilon) \leq 0.$$

Combining with (7.23), we then get

$$\begin{aligned}
\eta &\leq r_\varepsilon - r'_\varepsilon + b(x_\varepsilon)q_\varepsilon - b(x'_\varepsilon)q'_\varepsilon\\
&\quad + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(x_\varepsilon)A_\varepsilon - \sigma\sigma'(x'_\varepsilon)A'_\varepsilon\right) + f(x_\varepsilon) - f(x'_\varepsilon).
\end{aligned} \tag{7.24}$$

We now analyze the convergence of the r.h.s. of (7.24) as ε goes to zero. First, we see from (7.13) and (7.17) and (7.18) that $r_\varepsilon - r'_\varepsilon$ converge to zero. We also immediately see from the continuity of f and (7.13) that $f(x_\varepsilon) - f(x'_\varepsilon)$ converge to zero. It is also clear from the Lipschitz property of b, (7.13), (7.15), and (7.19) and (7.20) that $b(x_\varepsilon)q_\varepsilon - b(x'_\varepsilon)q'_\varepsilon$ converge to zero. Finally, from (7.21), we have

$$\begin{aligned}
\mathrm{tr}\left(\sigma\sigma'(x_\varepsilon)A_\varepsilon - \sigma\sigma'(x'_\varepsilon)A'_\varepsilon\right) &\leq \frac{3}{\varepsilon}\mathrm{tr}\left((\sigma(x_\varepsilon) - \sigma(x'_\varepsilon))(\sigma(x_\varepsilon) - \sigma(x'_\varepsilon))'\right)\\
&\quad - \mathrm{tr}\left(\sigma\sigma'(x_\varepsilon)Q(x_\varepsilon - x_0)\right),
\end{aligned}$$

and the r.h.s. of the above inequality converges to zero from the Lipschitz property of σ, (7.13) and (7.15). Therefore, by sending ε to zero into (7.24), we obtain the required contradiction: $\eta \leq 0$.

► *Cases* 5 *and* 6: We only consider the proof of case 5, as case 6 is similar. We keep the same notations as in the previous case. The crucial difference is that $\overline{u_{k_0}}$ and $\underline{w_{k_0}}$ may be sub and supersolution to different equations, depending on the position of t_ε (resp. t'_ε) with respect to $t^\varepsilon_{k_0} + h$ (resp. $t'^\varepsilon_{k_0} + h$). Actually, up to a subsequence for ε, we have three subcases. If $t_\varepsilon \geq t^\varepsilon_{k_0} + h$ and $t'_\varepsilon \geq t'^\varepsilon_{k_0} + h$ for all ε small enough, the proof of the preceding case applies. If $t_\varepsilon < t^\varepsilon_{k_0} + h$, for all ε small enough, then we have the viscosity subsolution (resp. supersolution) property of $\overline{u_{k_0}}$ (resp. $\underline{w_{k_0}}$) to the same linear PDE: $-\frac{\partial v_k}{\partial t} - \mathcal{L}v_k - f = 0$, at $(t_\varepsilon, x_\varepsilon, p_\varepsilon)$ (resp. $(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon)$), and we conclude as in *Case* 3. Finally, if $t_\varepsilon \geq t^\varepsilon_{k_0} + h$ and $t'_\varepsilon < t'^\varepsilon_{k_0} + h$ for all ε small enough, then the viscosity subsolution property of u_{k_0} to (4.2) at $(t_\varepsilon, x_\varepsilon, p_\varepsilon)$, and the viscosity η-strict supersolution property of w_{k_0} to (4.1) at $(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon)$ lead to:

$$-r'_\varepsilon - b(x'_\varepsilon)q'_\varepsilon - \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(x'_\varepsilon)A'_\varepsilon\right) - f(x'_\varepsilon) \geq \eta \tag{7.25}$$

and the following two possibilities:

$$-r_\varepsilon - b(x_\varepsilon)q_\varepsilon - \frac{1}{2}\mathrm{tr}\left(\sigma\sigma'(x_\varepsilon)A_\varepsilon\right) - f(x_\varepsilon) \leq 0, \tag{7.26}$$

or

$$\overline{u_{k_0}}(t_\varepsilon, x_\varepsilon, p_\varepsilon) - \sup_{e\in E} \overline{u_{k_0+1}}(t_\varepsilon, x_\varepsilon, p_\varepsilon \cup (t_\varepsilon, e)) \leq 0. \tag{7.27}$$

The first possibility (7.25), (7.26) is dealt with by the same arguments as in *Case* 4 (ii). The second possibility (7.25), (7.27) does not allow us to conclude directly. In fact, we use the additional condition (7.10):

$$\underline{w_{k_0}}(t_0, x_0, p_0) \geq \sup_{e\in E} \underline{w_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)) + \eta. \tag{7.28}$$

Since $\underline{w_{k_0}}$ is lower-semicontinuous, this implies by (7.13) that for all ε small enough:

$$\underline{w_{k_0}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon) \geq \underline{w_{k_0}}(t_0, x_0, p_0) - \frac{\eta}{2} \geq \sup_{e\in E} \underline{w_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)) + \frac{\eta}{2}.$$

Hence, by combining with (7.27), we deduce that

$$\begin{aligned}
&\overline{u_{k_0}}(t_\varepsilon, x_\varepsilon, p_\varepsilon) - \underline{w_{k_0}}(t'_\varepsilon, x'_\varepsilon, p'_\varepsilon) + \frac{\eta}{2} \\
&\quad\leq \sup_{e\in E} \overline{u_{k_0+1}}(t_\varepsilon, x_\varepsilon, p_\varepsilon \cup (t_\varepsilon, e)) - \sup_{e\in E} \underline{w_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)),
\end{aligned}$$

for all ε small enough. From (7.14) and Lemma 6.1, we then obtain by sending ε to zero:

$$\begin{aligned}
&\overline{u_{k_0}}(t_0, x_0, p_0) - \underline{w_{k_0}}(t_0, x_0, p_0) + \frac{\eta}{2} \\
&\quad\leq \sup_{e\in E} \overline{u_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)) - \sup_{e\in E} \underline{w_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)) \\
&\quad\leq \sup_{e\in E} \left\{\overline{u_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e)) - \underline{w_{k_0+1}}(t_0, x_0, p_0 \cup (t_0, e))\right\}.
\end{aligned}$$

This is in contradiction with (7.12). □

Finally, as usual, the comparison theorem for strict supersolutions implies comparison for supersolutions.

Proof of Proposition 7.2. For any $\eta > 0$, we use Lemma 7.4 to obtain an η-strict supersolution w_k^η of (4.1) and (4.2), which satisfies (7.2), so that $\underline{w_k}(t, x, p) \to w_k^\eta(t, x, p)$ for all $(t, x, p) \in \mathcal{D}_k$, as η goes to zero. We then use Lemma 7.5 to deduce that $\overline{u_k} \leq \underline{w_k}^\eta$ on $\mathcal{D}_k(n)$, $k = 0, \ldots, (n-m)\wedge m$. Thus, letting $\eta \to 0$, completes the proof. □

7.2. Boundary data and continuity

In this paragraph, we shall derive by induction the boundary data (4.4) and (4.5) in Proposition 4.3, and the continuity of the value functions as byproducts of viscosity properties and sequential comparison principles.

We first show relation (4.5), which follows easily from the definition of the value functions.

Lemma 7.6. *For all $x \in \mathbb{R}^d$, $v_0(T^-, x)$ exists and is equal to:*

$$v_0(T^-, x) = g(x). \tag{7.29}$$

Proof. For any $(t, x) \in (T - mh, T) \times \mathbb{R}^d$, we have from the definition of v_0, and the fact that no order can be passed after $T - mh$:

$$v_0(t, x) = \mathbb{E}\left[\int_t^T f(X_s^{t,x,0})\mathrm{d}s + g(X_T^{t,x,0})\right].$$

Therefore, with the continuity and linear growth assumptions on f and g, we get the result from the dominated convergence theorem. □

The derivation of relation (4.4) is more delicate. We first state the following result, which is a direct consequence of the dynamic programming principle.

Lemma 7.7. (i) *For $k = 1, \ldots, m$, and $p = (t_i, e_i)_{1 \le i \le k} \in \Theta_k \times E^k$, we have for all $x \in \mathbb{R}^d$, and $t \in [t_k, (t_k + h) \wedge (t_1 + mh))$,*

$$\begin{aligned} v_k(t, x, p) = \mathbb{E}\Bigg[&\int_t^{(t_k+h)\wedge(t_1+mh)} f(X_s^{t,x,0})\mathrm{d}s + v_k(t_k + h, X_{t_k+h}^{t,x,0}, p)1_{t_k+h<t_1+mh} \\ &+ \left(c(X_{t_1+mh}^{t,x,0}, e_1) + v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\right) 1_{t_1+mh\le t_k+h}\Bigg]. \end{aligned} \tag{7.30}$$

(ii) *For $k = 1, \ldots, m$, and $p = (t_i, e_i)_{1 \le i \le k} \in \Theta_k \times E^k$, we have for all $x \in \mathbb{R}^d$, and $t \in (T - mh, T)$,*

$$\begin{aligned} v_k(t, x, p) = \mathbb{E}\Bigg[&\int_t^{t_1+mh} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t_1+mh}^{t,x,0}, e_1) \\ &+ v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\Bigg]. \end{aligned} \tag{7.31}$$

(iii) *For $k = 1, \ldots, m$, and $p = (t_i, e_i)_{1 \le i \le k} \in \Theta_k \times E^k$, such that $t_k + h < t_1 + mh$ and $t_k + h \le T - mh$, we have for all $x \in \mathbb{R}^d$, and $t \in \mathbb{T}_p^2(k) = [t_k + h, t_1 + mh) \cap [0, T - mh]$,*

$$\begin{aligned} v_k(t, x, p) \ge \mathbb{E}\Bigg[&\int_t^{t_1+mh} f(X_s^{t,x,0})\mathrm{d}s \\ &+ c(X_{t_1+mh}^{t,x,0}, e_1) + v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\Bigg] \end{aligned} \tag{7.32}$$

$$\begin{aligned} v_k(t, x, p) \le \sup_{(\tau,\xi)\in\mathcal{I}_t} \mathbb{E}\Bigg[&\int_t^{(t_1+mh)\wedge\tau} f(X_s^{t,x,0})\mathrm{d}s + v_{k+1}(\tau, X_\tau^{t,x,0}, p \cup (\tau, \xi))1_{\tau<t_1+mh} \\ &+ \left(c(X_{t_1+mh}^{t,x,0}, e_1) + v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\right) 1_{t_1+mh\le\tau}\Bigg]. \end{aligned} \tag{7.33}$$

Proof. First, we recall from the dynamic programming principle that by making an immediate impulse control, i.e. by taking in (3.8), $\theta = t$ and $\tau = t$, $\xi = e$ arbitrary in E, we have for all $k = 0, \dots, m-1$, $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k \times E^k$, $(t, x) \in \mathbb{T}_p(k) \times \mathbb{R}^d$ with $t \ge t_k + h$,

$$v_k(t, x, p) \ge \sup_{e\in E} v_{k+1}(t, x, p \cup (t, e)). \tag{7.34}$$

(i) Fix $k = 1, \dots, m$, $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k \times E^k$, and $(t, x) \in \mathbb{T}^1_p(k) \times \mathbb{R}^d$. We distinguish the two following cases:

• *Case* 1: $t_k + h < t_1 + mh$. Then, for all $\alpha \in \mathcal{A}_{t,p}$, we have from (2.3), $X_s^{t,x,p,\alpha} = X_s^{t,x,0}$ for $t \le s \le t_k + h$. Hence, by applying (3.4) with $\theta = t_k + h$, and noting that $\tau_i + mh > \theta$, $k(\theta, \alpha) = k$, $p(\theta, \alpha) = p$ for any $\alpha = (\tau_i, \xi_i) \in \mathcal{A}_{t,p}$, we obtain the required relation (7.30), i.e.

$$v_k(t, x, p) = \mathbb{E}\left[\int_t^{t_k+h} f(X_s^{t,x,0})\mathrm{d}s + v_k(t_k + h, X_{t_k+h}^{t,x,0}, p)\right].$$

• *Case* 2: $t_1 + mh \le t_k + h$. Then, for all $\alpha \in \mathcal{A}_{t,p}$, we have from (2.3), $X_s^{t,x,p,\alpha} = X_s^{t,x,0}$ for $t \le s < t_1 + mh$, and $X_{t_1+mh}^{t,x,p,\alpha} = \Gamma(X_{t_1+mh}^{t,x,0}, e_1)$. Hence, by applying (3.4) with $\theta = t_1 + mh$, and noting that for any $\alpha = (\tau_i, \xi_i) \in \mathcal{A}_{t,p}$, we have either $k(\theta, \alpha) = k - 1$, $p(\theta, \alpha) = p_-$ if $\tau_{k+1} > t_1 + mh$ (which always arises when $t_1 + mh < t_k + h$), or $k(\theta, \alpha) = k$, $p(\theta, \alpha) = p_- \cup (\tau_{k+1}, \xi_{k+1})$ if $\tau_{k+1} = t_k + h = t_1 + mh$, we obtain

$$\begin{aligned} v_k(t, x, p) = \sup_{\alpha\in\mathcal{A}_{t,p}} \mathbb{E}\bigg[&\int_t^{t_1+mh} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t_1+mh}^{t,x,0}, e_1) \\ &+ v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)1_{\tau_{k+1}>t_1+mh} \\ &+ v_k(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_- \cup (t_1 + mh, \xi_{k+1}))1_{\tau_{k+1}=t_1+mh=t_k+h}\bigg]. \end{aligned}$$

Now, from (7.34), if $t_1+mh = t_k+h$, we have $v_k(t_1+mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-\cup(t_1+mh, \xi_{k+1})) \le v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)$ for all $\xi_{k+1}\mathcal{F}_{t_1+mh}$-measurable valued in E. We then deduce

$$\begin{aligned} v_k(t, x, p) = \mathbb{E}\bigg[&\int_t^{t_1+mh} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t_1+mh}^{t,x,0}, e_1) \\ &+ v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)\bigg], \end{aligned}$$

which is the required relation (7.30).

(ii) The proof is analogous to (i), case 1, as if $\tau_i > t - mh$, then $\tau_i = +\infty$.

(iii) Fix $k = 1, \dots, m$, $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k \times E^k$, s.t. $t_k + h < t_1 + mh$, and $(t, x) \in \mathbb{T}^2_p(k) \times \mathbb{R}^d$. Then, for all $\alpha \in \mathcal{A}_{t,p}$, we have from (2.3), $X_s^{t,x,p,\alpha} = X_s^{t,x,0}$ for $t \le s < t_1 + mh$, and $X_{t_1+mh}^{t,x,p,\alpha} = \Gamma(X_{t_1+mh}^{t,x,0}, e_1)$. Let $\alpha = (\tau_i, \xi_i)$ be some arbitrary element in $\mathcal{A}_{t,p}$, and set $\tau = \tau_{k+1}$, $\xi = \xi_{k+1}$. Observe that with $\theta = (t_1 + mh) \wedge \tau$, we have a.s. either $k(\theta, \alpha) = k + 1$, $p(\theta, \alpha) = p \cup (\tau, \xi)$ if $\tau < t_1 + mh$ or $k(\theta, \alpha) = k - 1$, $p(\theta, \alpha) = p_-$ if $\tau > t_1 + mh$, or $k(\theta, \alpha) = k$, $p(\theta, \alpha) = p_- \cup (\tau, \xi)$ if $\tau = t_1 + mh$. Hence, by applying (3.5) to some $\alpha = (\tau_i, \xi_i) \in \mathcal{A}_{t,p}$ s.t. $\tau_{k+1} > t_1 + mh$ a.s. and with $\theta = t_1 + mh$, we get the inequality (7.32). Furthermore, from (3.6), for all $\varepsilon > 0$, there exists $\alpha = (\tau_i, \xi_i) \in \mathcal{A}_{t,p}$ s.t. by setting $\tau = \tau_{k+1}$,

$\xi = \xi_{k+1}$, and with $\theta = (t_1 + mh) \wedge \tau$,

$$
\begin{aligned}
v_k(t, x, p) - \varepsilon \le \mathbb{E}\Bigg[& \int_t^{(t_1+mh)\wedge\tau} f(X_s^{t,x,0})\mathrm{d}s + v_{k+1}(\tau, X_\tau^{t,x,0}, p \cup (\tau, \xi))1_{\tau < t_1+mh} \\
& + c(X_{t_1+mh}^{t,x,0}, e_1)1_{t_1+mh \le \tau} + v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)1_{t_1+mh<\tau} \\
& + \ v_k(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_- \cup (\tau, \xi))1_{\tau=t_1+mh} \Bigg].
\end{aligned}
$$

Now, we have $v_k(t_1+mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-\cup(t_1+mh, \xi)) \le v_{k-1}(t_1+mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-)$ from (7.34). Since $(\tau, \xi) \in \mathcal{I}_t$, and ε is arbitrary, we deduce the required relation (7.33). $\square$

Proposition 7.8. *For all $k = 0, \ldots, m$, v_k is continuous on $\mathcal{D}_k$. Moreover, for all $k = 1, \ldots, m$, $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k \times E^k$, $x \in \mathbb{R}^d$,*

$$v_k((t_1 + mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-).$$

Proof. We shall prove by forward induction on $n = m, \ldots, N$ that (Hk)(n), $k = 1, \ldots, m(n)$, and (H0)(n) hold, where

(Hk)(n) v_k is continuous on $\mathcal{D}_k(n)$, and for all $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k(n) \times E^k$, $v_k((t_1 + mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-)$, $x \in \mathbb{R}^d$.

(H0)(n) v_0 is continuous on $\mathcal{D}_0(n)$,

with the convention that (Hk)(n) is empty for $n = m$.

▶ **Initialization**: $n = m$. We know from Proposition 4.2 that v_0 is a viscosity solution to (4.1) and (4.2) at step m. From Lemma 7.6 we get $\overline{v_0}(T, x) = \underline{v_0}(T, x) = g(x)$. Together with the comparison principle at step $n = m$ in Proposition 7.2, we get $\overline{v_0} \le \underline{v_0}$ on $\mathcal{D}_0(m)$. This implies continuity of v_0 on $\mathcal{D}_0(m)$, i.e. (H0)(m) is satisfied.

▶ **Step** $n \to n + 1$: $n \in \{m, \ldots, N - 1\}$. Suppose that (Hk)(n), $k = 1, \ldots, m(n)$, and (H0)(n) hold. Let us prove that (Hk)(n + 1), $k = 1, \ldots, m(n + 1)$, and (H0)(n + 1) are satisfied.

• Take some $k = 1, \ldots, m(n + 1)$, and fix some arbitrary $x \in \mathbb{R}^d$ and $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k(n + 1) \times \mathbb{E}^k$. Notice that $p_- \in \Theta_{k-1}(n) \times E^{k-1}$ so that $v_{k-1}(., ., p_-)$ is continuous on $\mathbb{T}_{p_-}(k - 1) \times \mathbb{R}^d$ from step n. Here, to alleviate notations, we used the convention that $\mathbb{T}_{p_-}(k - 1) = \mathbb{T}^n(0)$ if $k - 1 = 0$. We distinguish two cases:

⋆ *Case* 1. For some $\varepsilon > 0$, $\mathbb{T}_p^2(k) \cap [t_1 + mh - \varepsilon, t_1 + mh) = \emptyset$, i.e. $t_1 + mh \le t_k + h$ or $T - mh < t_1 + mh$ so that $[t_1 + mh - \varepsilon, t_1 + mh) \in \mathbb{T}_p^1(k)$. From (7.30) and (7.31), we then have for all $t \in [t_1 + mh - \varepsilon, t_1 + mh)$:

$$
\begin{aligned}
v_k(t, x, p) = \mathbb{E}\Bigg[& \int_t^{t_1+mh} f(X_s^{t,x,0})\mathrm{d}s + c(X_{t_1+mh}^{t,x,0}, e_1) \\
& + \ v_{k-1}(t_1 + mh, \Gamma(X_{t_1+mh}^{t,x,0}, e_1), p_-) \Bigg].
\end{aligned}
$$

By continuity of $v_{k-1}(t_1 + mh, ., p_-)$ (proved at step n), $\Gamma(., e_1)$, $c(., e_1)$, growth condition on f, c, Γ and v_{k-1}, we deduce with the dominated convergence theorem that $v_k((t_1 + mh)^-, x, p)$ exists and

$$v_k((t_1 + mh)^-, x, p) = c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-).$$

⋆ *Case* 2. $\mathbb{T}_p^2(k) = [t_k + h, t_1 + mh) \neq \emptyset$, i.e. $T - mh \geq t_1 + mh > t_k + h$ (this implies in particular that $k < (n + 1 - m) \wedge m$ and $m > 1$). From (7.32) and (7.33), we first prove that

$$\overline{v_k}(t_1 + mh, x, p) \leq \max\Bigg[c(x, e_1) + v_{k-1}(t_1 + mh, x, p_-), \sup_{e\in E} \overline{v_{k+1}}(t_1 + mh, x, p \cup (t_1 + mh, e))\Bigg]. \tag{7.35}$$

Indeed, consider some sequence $(t_\varepsilon, x_\varepsilon, p_\varepsilon)_{\varepsilon>0} \in \mathcal{D}_k$ converging to $(t_1 + mh, x, p)$ and such that $\lim_{\varepsilon\to 0} v_k(t_\varepsilon, x_\varepsilon, p_\varepsilon) = \overline{v_k}(t_1 + mh, x, p)$. For any $\varepsilon > 0$, one can find, by (7.33), some $(\hat{\tau}_\varepsilon, \hat{\xi}_\varepsilon) \in \mathcal{I}_{t_\varepsilon}$ s.t.

$$\begin{aligned} v_k(t_\varepsilon, x_\varepsilon, p_\varepsilon) \leq \mathbb{E}\Bigg[&\int_{t_\varepsilon}^{(t_1^\varepsilon+mh)\wedge\hat{\tau}_\varepsilon} f(X_s^{t_\varepsilon,x_\varepsilon,0})\mathrm{d}s \\ &+ v_{k+1}(\hat{\tau}_\varepsilon, X_{\hat{\tau}_\varepsilon}^{t_\varepsilon,x_\varepsilon,0}, p_\varepsilon \cup (\hat{\tau}_\varepsilon, \hat{\xi}_\varepsilon))1_{\hat{\tau}_\varepsilon<t_1^\varepsilon+mh} \\ &+ \left(c(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0}, e_1^\varepsilon) + v_{k-1}(t_1^\varepsilon + mh, \Gamma(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0}, e_1^\varepsilon), p_{\varepsilon-})\right) 1_{t_1^\varepsilon+mh\leq\hat{\tau}_\varepsilon}\Bigg] + \varepsilon, \end{aligned}$$

where we denote $p_\varepsilon = (t_i^\varepsilon, e_i^\varepsilon)_{1\leq i\leq k}$ and $p_{\varepsilon-} = (t_i^\varepsilon, e_i^\varepsilon)_{2\leq i\leq k}$. By setting

$$G_\varepsilon = c(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0}, e_1^\varepsilon) + v_{k-1}(t_1^\varepsilon + mh, \Gamma(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0}, e_1^\varepsilon), p_{\varepsilon-}),$$

we rewrite the above inequality as

$$\begin{aligned} v_k(t_\varepsilon, x_\varepsilon, p_\varepsilon) \leq \mathbb{E}\Bigg[&\int_{t_\varepsilon}^{(t_1^\varepsilon+mh)\wedge\hat{\tau}_\varepsilon} f(X_s^{t_\varepsilon,x_\varepsilon,0})\mathrm{d}s + G_\varepsilon \\ &+ \left(v_{k+1}(\hat{\tau}_\varepsilon, X_{\hat{\tau}_\varepsilon}^{t_\varepsilon,x_\varepsilon,0}, p_\varepsilon \cup (\hat{\tau}_\varepsilon, \hat{\xi}_\varepsilon)) - G_\varepsilon\right) 1_{\hat{\tau}_\varepsilon<t_1^\varepsilon+mh}\Bigg] + \varepsilon. \end{aligned} \tag{7.36}$$

Since $p_- \in \Theta_{k-1}(n) \times E^{k-1}$, we have $p_{\varepsilon-} \in \Theta_{k-1}(n) \times E^{k-1}$ for ε small enough. Hence, by continuity of v_{k-1} on $\mathcal{D}_{k-1}(n)$ (from part **1**.), continuity of Γ and c, and path-continuity of the flow $X_s^{t,x,0}$, we have

$$\lim_{\varepsilon\to 0} G_\varepsilon = G := c(x, e_1) + v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-) \quad a.s. \tag{7.37}$$

Moreover, by compactness of E, the sequence $(\hat{\xi}_\varepsilon)_\varepsilon$ converges, up to a subsequence, to some ξ valued in E. We deduce that

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \left(v_{k+1}(\hat{\tau}_\varepsilon, X_{\hat{\tau}_\varepsilon}^{t_\varepsilon,x_\varepsilon,0}, p_\varepsilon \cup (\hat{\tau}_\varepsilon, \hat{\xi}_\varepsilon)) - G_\varepsilon\right) 1_{\hat{\tau}_\varepsilon<t_1^\varepsilon+mh} \\ &\quad\leq (\overline{v_{k+1}}(t_1 + mh, x, p \cup (t_1 + mh, \xi)) - G) \limsup_{\varepsilon\to 0} 1_{\hat{\tau}_\varepsilon<t_1^\varepsilon+mh} \\ &\quad\leq \left(\sup_{e\in E} \overline{v_{k+1}}(t_1 + mh, x, p \cup (t_1 + mh, e)) - G\right) \limsup_{\varepsilon\to 0} 1_{\hat{\tau}_\varepsilon<t_1^\varepsilon+mh} \quad a.s. \end{aligned} \tag{7.38}$$

From the linear growth condition on f, c, Γ, v_{k-1}, v_{k+1}, and estimate (2.8), we may use dominated convergence theorem and send ε to zero in (7.36) to obtain with (7.37) and (7.38):

$$\begin{aligned}&\overline{v_k}(t_1+mh,x,p)\\&\quad\leq \mathbb{E}\left[G+\left(\sup_{e\in E}\overline{v_{k+1}}(t_1+mh,x,p\cup(t_1+mh,e))-G\right)\limsup_{\varepsilon\to 0}1_{\hat{\tau}_\varepsilon<t_1^\varepsilon+mh}\right]\\&\quad\leq \max\left[G,\sup_{e\in E}\overline{v_{k+1}}(t_1+mh,x,p\cup(t_1+mh,e))\right],\end{aligned}$$

which is the required inequality (7.35).

We next show that

$$\sup_{e\in E}\overline{v_{k+1}}(t_1+mh,x,p\cup(t_1+mh,e))\leq c(x,e_1)+v_{k-1}(t_1+mh,x,p_-). \tag{7.39}$$

Indeed, for any arbitrary $e\in E$, consider some sequence $(t_\varepsilon,x_\varepsilon,p_\varepsilon,e_\varepsilon)_{\varepsilon>0}\in\mathcal{D}_k\times E$ converging to (t_1+mh,x,p,e) and such that $\lim_{\varepsilon\to 0}v_{k+1}(t_\varepsilon,x_\varepsilon,p_\varepsilon\cup(t_\varepsilon,e_\varepsilon))=\overline{v_{k+1}}(t_1+mh,x,p\cup(t_1+mh,e))$. For ε small enough, $t_\varepsilon+h\geq t_1^\varepsilon+mh$, and so from the DPP (7.30), we have:

$$\begin{aligned}v_{k+1}(t_\varepsilon,x_\varepsilon,p_\varepsilon\cup(t_\varepsilon,e_\varepsilon))&=\mathbb{E}\left[\int_{t_\varepsilon}^{t_1^\varepsilon+mh}f(X_s^{t_\varepsilon,x_\varepsilon,0})\mathrm{d}s+c(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0},e_1^\varepsilon)\right.\\&\quad\left.+\ v_k(t_1^\varepsilon+mh,\Gamma(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0},e_1^\varepsilon),p_{\varepsilon-}\cup(t_\varepsilon,e_\varepsilon))\right].\end{aligned}\tag{7.40}$$

Since $p_-\in\Theta_{k-1}(n)\times E^{k-1}$, we have $p_{\varepsilon-}\in\Theta_{k-1}(n)\times E^{k-1}$ for ε small enough. Hence, by continuity of v_k on $\mathcal{D}_k(n)$, continuity and growth linear condition of f, Γ and c, and path-continuity of the flow $X_s^{t,x,0}$, we send ε to zero in (7.40) and get by the dominated convergence theorem

$$\begin{aligned}&\overline{v_{k+1}}(t_1+mh,x,p\cup(t_1+mh,e))\\&\quad=c(x,e_1)+v_k(t_1+mh,\Gamma(x,e_1),p_-\cup(t_1+mh,e)).\end{aligned}\tag{7.41}$$

Moreover, from (7.34), we have $v_{k-1}(t_1+mh,\Gamma(x,e_1),p_-)\geq v_k(t_1+mh,\Gamma(x,e_1),p_-\cup(t_1+mh,e))$ for all $e\in E$. Plugging into (7.41), this proves (7.39).

Finally, we easily see from (7.32) that

$$\underline{v_k}(t_1+mh,x,p)\geq c(x,e_1)+v_{k-1}(t_1+mh,x,p_-). \tag{7.42}$$

Indeed, consider some sequence $(t_\varepsilon,x_\varepsilon,p_\varepsilon)_{\varepsilon>0}\in\mathcal{D}_k$ converging to (t_1+mh,x,p) and such that $\lim_{\varepsilon\to 0}v_k(t_\varepsilon,x_\varepsilon,p_\varepsilon)=\underline{v_k}(t_1+mh,x,p)$. From (7.32), we have in particular

$$\begin{aligned}v_k(t_\varepsilon,x_\varepsilon,p_\varepsilon)&\geq\mathbb{E}\left[\int_{t_\varepsilon}^{t_1^\varepsilon+mh}f(X_s^{t_\varepsilon,x_\varepsilon,0})\mathrm{d}s\right.\\&\quad\left.+\ c(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0},e_1^\varepsilon)+v_{k-1}(t_1^\varepsilon+mh,\Gamma(X_{t_1^\varepsilon+mh}^{t_\varepsilon,x_\varepsilon,0},e_1^\varepsilon),p_{\varepsilon-})\right].\end{aligned}$$

By continuity and linear growth condition of v_{k-1}, Γ, c, f, and estimate (2.8), we get (7.42) by the dominated convergence theorem, and sending ε to zero in the above inequality.

Hence, the inequalities (7.35), (7.39) and (7.42) prove that $v_k((t_1+mh)^-,x,p)$ exists and is equal to:

$$\begin{aligned}v_k((t_1+mh)^-,x,p)&=\overline{v_k}(t_1+mh,x,p)=\underline{v_k}(t_1+mh,x,p)\\&=c(x,e_1)+v_{k-1}(t_1+mh,\Gamma(x,e_1),p_-).\end{aligned}\tag{7.43}$$

We have then proved that (7.43) holds for all $k = 1, \ldots, m$, $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k(n+1) \times E^k$, and $x \in \mathbb{R}^d$.

• We know from Proposition 4.2 that the family of value functions v_k, $k = 0, \ldots, m(n+1)$, is a viscosity solution to (4.1) and (4.2), in particular at step $n+1$. We also recall from Lemma 7.6 that $\overline{v_0}(T, x) = \underline{v_0}(T, x) = g(x)$. Together with (7.43), and the comparison principle at step $n+1$ in Proposition 7.2, this proves $\overline{v_k} \le \underline{v_k}$ on $\mathcal{D}_k(n+1)$. This implies the continuity of v_k on $\mathcal{D}_k(n+1)$, $k = 0, \ldots, m(n+1)$, and so $(\overline{\mathrm{Hk}})(n+1)$, $k = 1, \ldots, m(n+1)$, and (H0)(n+1) are stated.

▶ The proof is completed at step N by recalling that $\Theta_k(N) = \Theta_k$, $\mathcal{D}_k(N) = \mathcal{D}_k$, for $k = 0, \ldots, m(N) = m$. □

7.3. *Proof of* Theorem 4.4

In view of the results proved in Section 6 and paragraph 7.2, it remains to prove the uniqueness result of Theorem 4.4. Let us then consider another family w_k, $k = 0, \ldots, m$ of viscosity solutions to (4.1) and (4.2), satisfying growth condition (2.9), and boundary data (4.4) and (4.5): for $k = 1, \ldots, m$, $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k \times E^k$, $x \in \mathbb{R}^d$,

$$w_k((t_1 + mh)^-, x, p) = c(x, e_1) + w_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-) \tag{7.44}$$

and

$$w_0(T^-, x) = g(x), \quad x \in \mathbb{R}^d. \tag{7.45}$$

We shall prove by forward induction on $n = m, \ldots, N$ that $v_k = w_k$ on $\mathcal{D}_k(n)$.

▶ **Initialization**: $n = m$. Relations (4.5), (7.45) and Proposition 7.2 at step $n = m$ show that $v_0 = w_0$ on $\mathcal{D}_0(m)$.

▶ **Step** $n \to n+1$. Suppose that $v_k = w_k$ on $\mathcal{D}_k(n)$, $k = 0, \ldots, m(n)$. For any $k \ge 1$, $p = (t_i, e_i)_{1\le i\le k} \in \Theta_k(n+1) \times E^k$, we notice that $p_- \in \Theta_{k-1}(n) \times E^{k-1}$. Hence $v_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-) = w_{k-1}(t_1 + mh, \Gamma(x, e_1), p_-)$, $x \in \mathbb{R}^d$, and so from (4.4), (7.44), we have

$$v_k((t_1 + mh)^-, x, p) = w_k((t_1 + mh)^-, x, p).$$

We already know that $v_0(T^-, x) = w_0(T^-, x)$ $(= g(x))$. Therefore, from the comparison principle at step $n+1$ in Proposition 7.2, we deduce that $u_k = w_k$ on $\mathcal{D}_k(n+1)$, $k = 0, \ldots, m(n)$. Finally, the proof is completed since $\mathcal{D}_k(N) = \mathcal{D}_k$.

References

[1] L. Alvarez, J. Keppo, The impact of delivery lags on irreversible investment under uncertainty, European Journal on Operational Research 136 (2002) 173–180.

[2] A. Bar-Ilan, W. Strange, Investment lags, American Economic Review 86 (1996) 610–622.

[3] A. Bar-Ilan, A. Sulem, Explicit solution of inventory problems with delivery lags, Mathematics of Operations Research 20 (1995) 709–720.

[4] E. Bayraktar, M. Egami, The effects of implementation delay on decision-making under uncertainty, Stochastic Processes and their Applications 117 (2007) 333–358.

[5] A. Bensoussan, J.L. Lions, Contrôle impulsionnel et inéquations variationnelles, Dunod, 1982.

[6] D. Bertsekas, S. Shreve, Stochastic optimal control: The discrete time case, in: Math. in Sci. and Eng., Academic Press, 1978.

[7] M. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bulletin of the American Mathematical Society 27 (1992) 1–67.
[8] N. El Karoui, Les aspects probabilistes du contrôle stochastique, in: Lect. Notes in Math., Springer Verlag, 1981.
[9] W. Fleming, M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer Verlag, 1993.
[10] V. Ly Vath, M. Mnif, H. Pham, A model of portfolio selection under liquidity risk and price impact, Finance and Stochastics 11 (2007) 51–90.
[11] B. Oksendal, A. Sulem, Applied Stochastic Control of Jump Diffusions, Springer Verlag, 2005.
[12] B. Oksendal, A. Sulem, Optimal stochastic impulse control with delayed reaction, Applied Mathematics and Optimization (2006) (in press).
[13] H. Pham, On some recent aspects of stochastic control and their applications, Probability Surveys 2 (2005) 506–549.
[14] M. Robin, Contrôle impulsionnel avec retard pour les processus de diffusion, C.R.A.S 282 (1976) 463–466.
[15] A. Subramanian, R. Jarrow, The liquidity discount, Mathematical Finance 11 (2001) 447–474.